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Superconvergence of Solution Derivatives for the Shortley–Weller Difference Approximation of Poisson's Equation. II. Singularity Problems

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ABSTRACT

This is a continued analysis on superconvergence of solution derivatives for the Shortley–Weller approximation in Li (Li, Z. C., Yamamoto, T., Fang, Q. (2003): Superconvergence of solution derivatives for the Shortley–Weller difference approximation of Poisson's equation, Part I. Smoothness problems. *J. Comp. and Appl. Math*. 152(2):307–333), which is to explore superconvergence for unbounded derivatives near the boundary. By using the stretching function proposed in Yamamoto (Yamamoto, T. (2002): Convergence of consistant and inconsistent finite difference schemes and an acceleration technique. *J. Comp. Appl. Math*. 140:849–866), the second order superconvergence for the solution derivatives can be established. Moreover, numerical experiments are provided to support the error analysis made. The analytical approaches in this article are non-trivial, intriguing, and different from Li, Z. C., Yamamoto, T., Fang, Q. (2003).

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This article also provides the superconvergence analysis for the bilinear finite element method and the finite difference method with nine nodes.

Key Words: Superconvergence; Solution derivatives; Boundary singularity; Finite difference method; Stretching function; The Shortley–Weller approximation; Poisson's equation.

AMS(MOS) Subject Classification: 65N10; 65N30.

1. INTRODUCTION

Recently, Yamamoto, etc. (Fang, 2001; Yamamoto, 2002; Yamamoto et al., 2001) considered unbounded derivatives of the solutions on $S = \{(x, y) \ |0 < x < 1, 0 < y < 1\}$ with $u \in O(x^\sigma(1-x)^\sigma)$ and $u \in O(y^\sigma(1-y)^\sigma)$, where $0 < \sigma < 2$. By means of a stretching function for local refinement difference grids, the superconvergence of the maximal nodal errors are achieved: $\max_{ij} |u_{ij} - (u_h)_{ij}| = O(h^2 \ln 1/h)$ by the Shortley–Weller approximation. In Yoshida and Tsuchiya (2001), an analysis is made for a posteriori derivatives from the nodal solutions obtained.

In this article, we continue the study in Li et al. (2003) for superconvergence of solution derivatives by the Shortley–Weller approximation. For the smooth solution $u \in C^3(\bar{S})$, the solution derivatives in a discrete H^1 norm are proved to be $O(h^2)$ in Li et al. (2003), where a brief view on literature of superconvergence is also given. This article is devoted to unbounded derivatives near the boundary, and the same superconvergence order $O(h^2)$ of solution derivatives in the discrete H^1 norm is developed for $u \in C^{1/2+\mu}(\bar{S})$ and $f \in C^{-3/2+\mu}(\bar{S})$, where $0 < \mu < 3/2$. The notation $u \in C^{1/2+\mu}(\bar{S}) (= C^{k,\alpha}(\bar{S}))$, where $(1/2) + \mu = k + \alpha$, k is an integer and $\alpha \in [0, 1)$. Also $C^{k,\alpha}(\bar{S})$ denotes the set of functions whose k-th derivatives are Hölder continuous in $\bar{S}$:

$$\sup_{\substack{(P,Q)\in\bar{S} \\ P\neq Q}} \frac{|D^k(P) - D^k(Q)|}{||P-Q||^\alpha} < \infty,$$

where $\|P\|$ is the Euclidean norm. Note that the conditions $u \in C^{1/2+\mu}(\bar{S})$, $0 < \mu < 3/2$, is necessary for error estimates of the solution derivatives from the analysis of the finite element method (FEM): $\|u - u_h\|_{1,S} \le h^\beta |u|_{1+\beta,S}, \beta > 0$. Such a condition is also valid for a posteriori derivatives from the nodal solutions of the Shortley–Weller approximation.

This article is organized as follows. In the next section, we interpret the Shortley–Weller difference approximation as a special FEM (Li, 1998; Li et al., 2003), and also discuss the bilinear FEM and the finite difference method (FDM) with nine nodes whose associated matrix is the same as that of the bilinear FEM. In Sec. 3, we derive superconvergence for the FDM with nine nodes, and in Sec. 4 for the Shortley–Weller approximation. In the last section, numerical experiments are provided to support the theoretical analysis made.

2. THE SHORTLEY–WELLER DIFFERENCE APPROXIMATION

Consider the Poisson equation with the Dirichlet boundary condition,

$$-\Delta u = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f(x,y), \ (x,y) \in S, \tag{2.1}$$

$$u = g(x,y), \ (x,y) \in \Gamma, \tag{2.2}$$

where S is a rectangle, Γ is the exterior boundary ∂S of S, and f and g are given functions. The domain S is again split by difference grids into small rectangles $\square_{ij}$ as shown in Fig. 1. Denote $u_{i,j} = u(x_i, y_j)$, where $(i,j)(= (x_i, y_j))$ denotes the location of difference nodes. Let $h_i = x_{i+1} - x_i$, $k_j = y_{j+1} - y_j$, and the maximal mesh size $h = \max_{i,j}\{h_i, k_j\}$. The conventional finite difference method can be regarded as a special kind of finite element methods using piecewise bilinear interpolatory functions $v(x,y)$ on $\square_{ij}$ (see Li, 1998; Li et al., 2003),

$$\begin{aligned} v(x,y) = \frac{1}{h_i k_j}\{&(x_{i+1} - x)(y_{j+1} - y)v_{ij} + (x - x_i)(y_{j+1} - y)v_{i+1,j} \\ &+(x_{i-1} - x)(y - y_j)v_{i,j+1} + (x - x_i)(y - y_j)v_{i+1,j+1}\}, \text{ for } (x,y) \in \square_{ij}. \end{aligned} \tag{2.3}$$

For ease of exploration, we let S be a rectangle. When S is a polygon, the extension of our proposed techniques will be briefly outlined in the last section of this article.

Let $V_h(\subseteq H^1(S))$ denote a finite dimensional collection of the piecewise bilinear functions v in Eq. (2.3) satisfying Eq. (2.2), and by V_h^0 those in Eq. (2.3) satisfying $v = 0, (x,y) \in \Gamma$. The Shortley–Weller approximation involving integral approximation on Γ can be expressed by: find $u_h \in V_h$ such that

$$\hat{a}_h(u_h, v) = \hat{f}_h(v), \quad \forall v \in V_h^0, \tag{2.4}$$

where

$$\hat{a}_h(u,v) = \widehat{\iint_S} \nabla u \nabla v \, ds, \tag{2.5}$$

and

$$\widehat{\iint_S} \nabla u \nabla v \, ds = \sum_{ij}\left[\widehat{\iint_{\square_{ij}}} \nabla u \nabla v \, ds\right], \tag{2.6}$$

$$\hat{f}_h(v) = \widehat{\iint_S} f \, v \, ds = \sum_{ij}\left[\widehat{\iint_{\square_{ij}}} f \, v \, ds\right]. \tag{2.7}$$

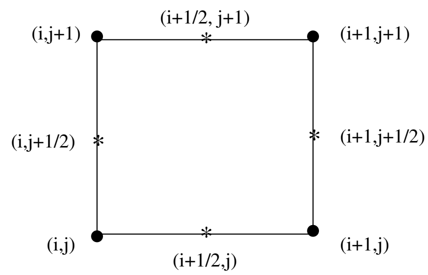

Figure 1. A rectangle $\square_{ij}$.

The approximate integrals in Eqs. (2.6) and (2.7) are evaluated by the following specific rules (see Fig. 1):

$$\widehat{\iint_{\square_{ij}}} \nabla u \nabla v \, ds = \widehat{\iint_{\square_{ij}}} u_x v_x \, ds + \widehat{\iint_{\square_{ij}}} u_y v_y \, ds, \tag{2.8}$$

$$\widehat{\iint_{\square_{ij}}} u_x v_x \, ds = \frac{h_i k_j}{2} \left[u_x(i+\frac{1}{2},j) v_x(i+\frac{1}{2},j) + u_x(i+\frac{1}{2},j+1) v_x(i+\frac{1}{2},j+1) \right], \tag{2.9}$$

$$\widehat{\iint_{\square_{ij}}} u_y v_y \, ds = \frac{h_i k_j}{2} \left[u_y(i,j+\frac{1}{2}) v_y(i,j+\frac{1}{2}) + u_y(i+1,j+\frac{1}{2}) v_y(i+1,j+\frac{1}{2}) \right], \tag{2.10}$$

$$\widehat{\iint_{\square_{ij}}} fv \, ds = \frac{h_i k_j}{4} \left[f_{ij} v_{ij} + f_{i+1,j} v_{i+1,j} + f_{i,j+1} v_{i,j+1} + f_{i+1,j+1} v_{i+1,j+1} \right], \tag{2.11}$$

where $u_x(i+1/2,j) = u_x(x_{i+1/2}, y_j)$, $u_y(i,j+1/2) = u_y(x_i, y_{j+1/2})$, $x_{i+1/2} = 1/2(x_i + x_{i+1})$ and $y_{j+1/2} = 1/2(y_j + y_{j+1})$. The finite difference equations at the interior nodes (i,j) are obtained from Eq. (2.4),

$$\begin{aligned} &-\frac{(k_{j-1}+k_j)}{2h_i}(u_{i+1,j}-u_{i,j}) - \frac{(k_{j-1}+k_j)}{2h_{i-1}}(u_{i-1,j}-u_{i,j}) \\ &-\frac{(h_{i-1}+h_i)}{2k_j}(u_{i,j+1}-u_{i,j}) - \frac{(h_{i-1}+h_i)}{2k_{j-1}}(u_{i,j-1}-u_{i,j}) \\ &\quad = \frac{(h_{i-1}+h_i)(k_{j-1}+k_j)}{4} f_{i,j}. \end{aligned} \tag{2.12}$$

Dividing two sides of Eq. (2.12) by $(h_{i-1}+h_i)(k_{j-1}+k_j)/4$ gives exactly the Shortley–Weller approximation in Bramble and Hubbard (1962),Yamamoto (2002).

The main objective of this article is to consider the case with $S = S_0 \cup S_\Gamma$, where $S_0 \subseteq S$ and $u \in C^3(\bar{S}_0)$. However, there exist unbounded derivatives in S_Γ near the boundary Γ. For example, let $a, b < 1/2$, two subdomains may be chosen as $S_0 = \{(x, y)|a < x < 1 - a, b < y < 1 - b\}$ and $S_\Gamma = S \backslash S_0$. We assume the following:

A1. The derivatives of the solution u satisfy the following properties (Fang, 2001; Yamamoto, 2002):

$$\sup_{x\in(0,1)} x^{i-\alpha_0}(1-x)^{i-\alpha_1}\left|\frac{\partial^i}{\partial x^i}u(x,y)\right| \le C_1, \quad i = 1, 2, 3, \tag{2.13}$$

$$\sup_{y\in(0,1)} y^{i-\beta_0}(1-y)^{i-\beta_1}\left|\frac{\partial^i}{\partial y^i}u(x,y)\right| \le C_2, \quad i = 1, 2, 3, \tag{2.14}$$

where C_1 and C_2 are constants. Moreover, the solution satisfies

$$\sup_{\mathrm{dist}(P,Q)\le r} |u(P) - u(Q)| \le Cr^\sigma, \tag{2.15}$$

where P and Q are near Γ, $\sigma = \min\{\alpha_0, \alpha_1, \beta_0, \beta_1\}$, and C is a constant independent of r and σ. In this article, we assume

$$\sigma = \frac{1}{2} + \mu, \quad \mu > 0, \tag{2.16}$$

which is necessary for the derivative estimates.

A2. The function f satisfies the following properties (Bramble and Hubbard, 1962; Fang, 2001):

$$\sup_{x\in(0,1)} x^{i-\alpha_2}(1-x)^{i-\alpha_3}\left|\frac{\partial^i}{\partial x^i}f(x,y)\right| \le C_3, \quad i = 0, 1, 2, \tag{2.17}$$

$$\sup_{y\in(0,1)} y^{i-\beta_2}(1-y)^{i-\beta_3}\left|\frac{\partial^i}{\partial y^i}f(x,y)\right| \le C_4, \quad i = 0, 1, 2, \tag{2.18}$$

where C_3 and C_4 are constants. We put $\nu = \min\{\alpha_2, \alpha_3, \beta_2, \beta_3\}$ and

$$\nu = -\frac{3}{2} + \mu, \quad \mu > 0, \tag{2.19}$$

which is compatible with Eq. (2.16).

A3. Let $[0, 1]$ be divided into a non-equidistant grids of local refinements near Γ,

$$x_i = \psi(ih), \quad y_j = \psi(jh), \quad i, j = 0, 1, .., n, \tag{2.20}$$

where $h = 1/n$ and the stretching function is given by Bramble and Hubbard (1962), Fang (2001):

$$\psi(s) = \frac{1}{\int_0^1 z^p(1-z)^p\, dz}\int_0^s z^p(1-z)^p\, dz, \tag{2.21}$$

and $p > 0$ is a suitable number. In S_Γ the difference grids (x_i, y_j) are given in Eq. (2.20).

Equation (2.21) is a kind of stretching functions to deal with singularities. An iterated SINE function is used in Tang and Trummer (1996), as another kind of stretching functions for singularly perturbed problems, and an error analysis is given in Liu and Tang (2001). For singularity problems, the traditional FEM and FDM fail to provide satisfactory solutions, because there occurs violation of the principle that difference quotients approximate derivatives, due to their unbounded values near the singularities. Therefore, local refinements of finite elements or difference grids near the singularities are inevitable, and stretching functions are, indeed, the detailed refinements. Of course, different stretching functions are effective to different singularity problems. In this article and Yamamoto (2002), the stretching function in **A3** is proven to be effective to the singularity problems satisfying **A1** and **A2**. On the other hand, the iterated SINE function is effective to singularly perturbed problems, see Liu and Tang (2001), Tang and Trummer, 1996).

In this article, we pursue superconvergence, based on the discrete H^1 norms:

$$\overline{\|v\|}_1^2 = \overline{\|v\|}_{1,S}^2 = \overline{|v|}_{1,S}^2 + \overline{\|v\|}_{0,S}^2, \tag{2.22}$$

$$\overline{|v|}_1^2 = \overline{|v|}_{1,S}^2 = \sum_{ij}\left[\widehat{\iint}_{\square_{ij}} (\nabla v)^2\, ds\right], \tag{2.23}$$

$$\overline{\|v\|}_0^2 = \overline{\|v\|}_{0,S}^2 = \sum_{ij}\left[\widehat{\iint}_{\square_{ij}} v^2\, ds\right]. \tag{2.24}$$

The discrete formulas, $\widehat{\iint}_{\square_{ij}} |\nabla v|^2\, ds$ and $\widehat{\iint}_{\square_{ij}} v^2\, ds$ are given by Eqs. (2.8)–(2.11). We can derive the estimates from Ciarlet (1991), p. 197,

$$\overline{\|u-u_h\|}_1 \le C\left\{\overline{\|u-u_I\|}_1 + \sup_{w\in V_h^0} \frac{|(\iint_S - \widehat{\iint}_S)\nabla u_I \nabla w\, ds|}{\overline{\|w\|}_1} + \sup_{w\in V_h^0} \frac{|(\iint_S - \widehat{\iint}_S) f\, w\, ds|}{\overline{\|w\|}_1}\right\}, \tag{2.25}$$

where u_I is the piecewise bilinear interpolant of the true solution u.

Let us consider three kinds of FEMs: To seek $u_h^E, u_h^D, u_h \in V_h$ such that

$$\iint_S \nabla u_h^E \nabla v = \iint f v, \quad \forall v \in V_h^0, \tag{2.26}$$

$$\iint_S \nabla u_h^D \nabla v = \widehat{\iint} f v, \quad \forall v \in V_h^0, \tag{2.27}$$

$$\widehat{\iint}_S \nabla u_h \nabla v = \widehat{\iint} f v, \quad \forall v \in V_h^0, \tag{2.28}$$

where u_h^E, u_h^D, u_h are the solutions from the bilinear FEM, the FDM with nine nodes and the Shortley–Weller method, respectively. A detailed discussion for Eqs. (2.26)–(2.28) is given in Li et al. (2003). For error analysis under unbounded derivatives near boundary, we will carry out two steps: first step for solution u_h^D from Eq. (2.27) in Sec. 3, and second step for solution u_h from Eq. (2.28) in Sec. 4. When there is no errors for integrals $\iint_S \nabla u \nabla v$ and $\iint_S fv\, ds$, we obtain the error analysis of u_h^E from the bilinear FEM. Hence, this article covers the analysis for all three kinds of FEMs in Eqs. (2.26)–(2.28).

3. ANALYSIS FOR u_h^D WITH NO ERRORS OF DIVERGENCE INTEGRATION

From Eq. (2.25), we have

$$\overline{\|u - u_h^D\|}_1 \leq C \left\{ \overline{\|u - u_I\|}_1 + \sup_{w \in V_h^0} \frac{|(\iint_S - \widehat{\iint}_S) f\, w\, ds|}{\overline{\|w\|}_1} \right\}. \tag{3.1}$$

Below, we will derive the bounds of two terms on the right hand side of Eq. (3.1). We first observe that, as was defined in Sec. 2, $S = S_0 \cup S_\Gamma$, where S_0 and $S_\Gamma = S \backslash S_0$ consist of small rectangles $\square_{ij}$. Then we define the norms

$$\overline{\|w\|}_{1, S_\Gamma}^2 = \overline{|w|}_{1, S_\Gamma}^2 + \overline{\|w\|}_{0, S_\Gamma}^2, \tag{3.2}$$

$$\overline{|w|}_{1, S_\Gamma}^2 = \sum_{\substack{\square_{ij} \in S_\Gamma \\ w \in V_h^0}} \left[\widehat{\iint}_{\square_{ij}} |\nabla w|^2\, ds \right], \quad \overline{\|w\|}_{0, S_\Gamma}^2 = \sum_{\substack{\square_{ij} \in S_\Gamma \\ w \in V_h^0}} \left[\widehat{\iint}_{\square_{ij}} w^2\, ds \right]. \tag{3.3}$$

The norms of $\overline{\|w\|}_{1, S_0}$ and $\overline{|w|}_{1, S_0}$ are also given by replacing S_Γ by S_0 in Eqs. (3.2) and (3.3). Then we have the following lemma.

Lemma 3.1. *Let u and $u_I \in V_h$ be the true solution and its piecewise bilinear interpolant of u on S_Γ, respectively. Then*

$$\overline{\|u - u_I\|}_{1, S_\Gamma} \leq \frac{1}{\sqrt{2}} \left\{ \sum_{\square_{ij} \in S_\Gamma} \iint_{\square_{ij}} \kappa_{1,i}^2(x) \big(u_{xxx}^2(x, y_j) + u_{xxx}^2(x, y_{j+1}) \big) + \sum_{\square_{ij} \in S_\Gamma} \iint_{\square_{ij}} \kappa_{2,j}^2(y) (u_{yyy}^2(x_i, y) + u_{yyy}^2(x_{i+1}, y)) \right\}^{1/2}, \tag{3.4}$$

where the functions are

$$\kappa_{1,i}(x) = \begin{cases} \frac{1}{2}(x - x_i)^2 & \text{for} \quad x_i \leq x \leq x_i + \frac{h_i}{2}, \\ \frac{1}{2}(x_{i+1} - x)^2 & \text{for} \quad x_i + \frac{h_i}{2} \leq x \leq x_{i+1}, \end{cases} \tag{3.5}$$

and

$$\kappa_{2,j}(y) = \begin{cases} \frac{1}{2}(y - y_j)^2 & for \quad y_j \le y \le y_j + \frac{k_j}{2}, \\ \frac{1}{2}(y_{j+1} - y)^2 & for \quad y_j + \frac{k_j}{2} \le y \le y_{j+1}. \end{cases} \tag{3.6}$$

Proof. Straight forward Taylor expansion gives

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0) + \frac{1}{2}\int_{x_0}^{x}(x - t)^2 f'''(t)\, dt. \tag{3.7}$$

Denote $w = u - u_I$, where u_I is the piecewise bilinear interpolant of u along x and y. We have

$$\begin{aligned} \left|w_x(i+\frac{1}{2},j)\right| &= \left|u_x(i+\frac{1}{2},j) - \frac{u(i+1,j) - u(i,j)}{h_i}\right| \\ &= \left|\frac{1}{h_i}\int_{x_i}^{x_{i+1}} \kappa_{1,i}(x)u_{xxx}(x,y_j)\, dx\right| \\ &\le \frac{1}{\sqrt{h_i}}\sqrt{\int_{x_i}^{x_{i+1}} \kappa_{1,i}^2(x)u_{xxx}^2(x,y_j)\, dx}, \end{aligned} \tag{3.8}$$

where we have used the Schwarz inequality. Similarly,

$$\begin{aligned} \left|w_y(i,j+\frac{1}{2})\right| &= \left|u_y(i,j+\frac{1}{2}) - \frac{u(i,j+1) - u(i,j)}{k_j}\right| \\ &\le \frac{1}{\sqrt{k_j}}\sqrt{\int_{y_j}^{y_{j+1}} \kappa_{2,j}^2(y)u_{yyy}^2(x_i,y)\, dy}. \end{aligned} \tag{3.9}$$

We then obtain

$$\begin{aligned} \overline{\|u-u_I\|}_{1,S_\Gamma}^2 &= \sum_{\square_{ij}\in S_\Gamma} \overline{|u-u_I|}_{1,\square_{ij}}^2 \\ &= \sum_{\square_{ij}\in S_\Gamma} \frac{h_i k_j}{2}\{w_x^2(i+\frac{1}{2},j) + w_x^2(i+\frac{1}{2},j+1) + w_y^2(i,j+\frac{1}{2}) + w_y^2(i+1,j+\frac{1}{2})\} \\ &\le \sum_{\square_{ij}\in S_\Gamma} \frac{h_i k_j}{2} \times \{\frac{1}{h_i}\int_{x_i}^{x_{i+1}} \kappa_{1,i}^2(x)(u_{xxx}^2(x,y_j) + u_{xxx}^2(x,y_{j+1})) \\ &\quad + \frac{1}{k_j}\int_{y_j}^{y_{j+1}} \kappa_{2,j}^2(y)(u_{yyy}^2(x_i,y) + u_{yyy}^2(x_{i+1},y))\} \\ &\le \frac{1}{\sqrt{2}}\sum_{\square_{ij}\in S_\Gamma} \iint_{\square_{ij}} \{\kappa_{1,i}^2(x)(u_{xxx}^2(x,y_j) + u_{xxx}^2(x,y_{j+1})) \\ &\quad + \kappa_{2,j}^2(y)(u_{yyy}^2(x_i,y) + u_{yyy}^2(x_{i+1},y))\}. \end{aligned} \tag{3.10}$$

This completes the proof of Lemma 3.1. ■

Lemma 3.2. *Let* $g \in C^2(\Box)$, *where* $\Box = \{(x, y)|0 \le x \le h, 0 \le y \le k\}$. *The trapezoidal rule on* $\Box$ *may be expressed as*

$$\iint_{\Box} g = \frac{hk}{4}\Big[g(0,0) + g(h,0) + g(0,k) + g(h,k)\Big] + R_T, \tag{3.11}$$

where the errors are given by

$$R_T = \frac{1}{4}\iint_{\Box}\Big\{\Big[g_{xx} + \frac{1}{2}(g_{xx}(x,0) + g_{xx}(x,k))\Big]x\,(x-h) + \Big[g_{yy} + \frac{1}{2}(g_{yy}(0,y) + g_{yy}(h,y))\Big]y\,(y-k)\Big\}. \tag{3.12}$$

Proof. From Atkinson (1989), p. 286, we have for the trapezoidal rule on $[0, h]$

$$\int_0^h g(x)\,dx = \frac{h}{2}(g(0) + g(h)) + \frac{1}{2}\int_0^h g_{xx}x(x-h)\,dx. \tag{3.13}$$

Hence, for the integration on $\Box$, there exists the equality,

$$\begin{aligned}\iint_{\Box} g(x,y) &= \int_0^k\Big\{\int_0^h g(x,y)\,dx\Big\}dy \\ &= \frac{h}{2}\int_0^k (g(0,y) + g(h,y))\,dy + \frac{1}{2}\int_0^k\int_0^h g_{xx}x(x-h)\,dx\,dy.\end{aligned} \tag{3.14}$$

Next from Eq. (3.13) again, we obtain

$$\int_0^k g(0,y)\,dy = \frac{k}{2}(g(0,0) + g(0,k)) + \frac{1}{2}\int_0^k g_{yy}(0,y)y\,(y-k)\,dy, \tag{3.15}$$

$$\int_0^k g(h,y)\,dy = \frac{k}{2}(g(h,0) + g(h,k)) + \frac{1}{2}\int_0^k g_{yy}(h,y)y\,(y-k)\,dy. \tag{3.16}$$

Combining Eqs. (3.14)–(3.16) gives

$$\begin{aligned}R_T &= \iint_{\Box} g(x,y) - \frac{hk}{4}(g(0,0) + g(h,0) + g(0,k) + g(h,k)) \\ &= \frac{1}{2}\iint_{\Box} g_{xx}x(x-h) + \frac{h}{4}\int_0^k (g_{yy}(0,y) + g_{yy}(h,y))y\,(y-k)\,dy \\ &= \frac{1}{2}\iint_{\Box} g_{xx}x(x-h) + \frac{1}{4}\iint_{\Box}(g_{yy}(0,y) + g_{yy}(h,y))y\,(y-k).\end{aligned} \tag{3.17}$$

From the symmetry between x and y, we also obtain

$$R_T = \frac{1}{2}\iint_{\Box} g_{yy}y\,(y-k) + \frac{1}{4}\iint_{\Box}(g_{xx}(x,0) + g_{xx}(x,k))x(x-h). \tag{3.18}$$

Combining Eqs. (3.17) and (3.18) gives the desired result (3.12). This completes the proof of Lemma 3.2. ■

Lemma 3.3. *There exists the bound*,

$$\left|\left(\iint_{S_\Gamma} - \widehat{\iint}_{S_\Gamma}\right) fw\right| \leq C\overline{|||f|||} \times \overline{\|w\|}_{1,S_\Gamma}, \quad w \in V_h^0, \tag{3.19}$$

where

$$\overline{|||f|||} = \sqrt{\sum_{\square_{ij}\in S_\Gamma}\left\{\widehat{\iint}_{\square_{ij}} p_{ij}^2 + \|q_{ij}\|_{-1,\square_{ij}}^2\right\}}, \tag{3.20}$$

and

$$\begin{aligned} p_{ij}^2 &= (f_x^2 + f_x^2(x, y_j) + f_x^2(x, y_{j+1}))(x - x_i)^2(x - x_{i+1})^2 \\ &\quad + (f_y^2 + f_y^2(x_i, y) + f_y^2(x_{i+1}, y))(y - y_j)^2(y - y_{j+1})^2, \end{aligned} \tag{3.21}$$

$$\begin{aligned} q_{ij}^2 &= (f_{xx}^2 + f_{xx}^2(x, y_j) + f_{xx}^2(x, y_{j+1}))(x - x_i)^2(x - x_{i+1})^2 \\ &\quad + (f_{yy}^2 + f_{yy}^2(x_i, y) + f_{yy}^2(x_{i+1}, y))(y - y_j)^2(y - y_{j+1})^2, \end{aligned} \tag{3.22}$$

where

$$\|q_{ij}\|_{-1,\square_{ij}} \leq \sup_{w\in V_h^0} \frac{\iint_{\square_{ij}} q_{ij} w}{\|w\|_{1,\square_{ij}}}. \tag{3.23}$$

Proof. Let $g = fw$, $w \in V_h^0$, then $g_{xx} = f_{xx}w + f_x w_x$ and $g_{yy} = f_{yy}w + f_y w_y$. We have from Lemma 3.2,

$$\begin{aligned} \left(\iint_{S_\Gamma} - \widehat{\iint}_{S_\Gamma}\right) fw &= \sum_{\square_{ij}\in S_\Gamma}\left(\iint_{\square_{ij}} - \widehat{\iint}_{\square_{ij}}\right) fw \\ &= \frac{1}{4}\sum_{\square_{ij}\in S_\Gamma}\iint_{\square_{ij}}\left\{(f_{xx}w + f_x w_x) + \frac{1}{2}(f_{xx}w + f_x w_x)(x, y_j)\right. \\ &\quad \left. + \frac{1}{2}(f_{xx}w + f_x w_x)(x, y_{j+1})\right\}(x - x_i)(x - x_{i+1}) \\ &\quad + \left\{(f_{yy}w + f_y w_y) + \frac{1}{2}(f_{yy}w + f_y w_y)(x_i, y)\right. \\ &\quad \left. + \frac{1}{2}(f_{yy}w + f_y w_y)(x_{i+1}, y)\right\}(y - y_j)(y - y_{j+1}). \end{aligned} \tag{3.24}$$

Denote $\delta_{1,i}(x)=(x-x_i)(x-x_{i+1})$. We have from the Schwarz inequality and Eq. (3.23),

$$\sum_{\square_{ij}\in S_\Gamma}\iint_{\square_{ij}}(f_{xx}w+f_xw_x)\delta_{1,i}(x)$$
$$\leq\sqrt{\sum_{\square_{ij}\in S_\Gamma}\iint_{\square_{ij}}f_x^2\delta_{1,i}^2(x)}\times\sqrt{\sum_{\square_{ij}\in S_\Gamma}\|w_x\|_{0,\square_{ij}}^2}$$
$$+\sqrt{\sum_{\square_{ij}\in S_\Gamma}\|f_{xx}\delta_{1,i}(x)\|_{-1,\square_{ij}}^2}\times\sqrt{\sum_{\square_{ij}\in S_\Gamma}\|w\|_{1,\square_{ij}}^2}$$
$$\leq\left\{\sqrt{\sum_{\square_{ij}\in S_\Gamma}\iint_{\square_{ij}}f_x^2\delta_{1,i}^2(x)}+\sqrt{\sum_{\square_{ij}\in S_\Gamma}\|f_{xx}\delta_{1,i}(x)\|_{-1,\square_{ij}}^2}\right\}\times\|w\|_{1,S_\Gamma}. \tag{3.25}$$

Similarly, we may have the bounds for other terms in Eq. (3.24), and then obtain

$$\left|\left(\iint_{S_\Gamma}-\widehat{\iint}_{S_\Gamma}\right)fw\right| \tag{3.26}$$
$$=\left\{\sum_{\square_{ij}\in S_\Gamma}\iint_{\square_{ij}}(f_x^2+f_x^2(x,y_j)+f_x^2(x,y_{j+1}))\delta_{1,i}^2(x)\right.$$
$$+\sum_{\square_{ij}\in S_\Gamma}\iint_{\square_{ij}}(f_y^2+f_y^2(x_i,y)+f_y^2(x_{i+1},y))\delta_{2,j}^2(y)$$
$$+\sum_{\square_{ij}\in S_\Gamma}\|(f_{xx}+f_{xx}(x,y_j)+f_{xx}(x,y_{j+1}))\delta_{1,i}(x)$$
$$\left.+(f_{yy}+f_{yy}(x_i,y)+f_{yy}(x_{i+1},y))\delta_{2,j}(y)\|_{-1,\square_{ij}}^2\right\}^{1/2}\times\|w\|_{1,S_\Gamma},$$

where $\delta_{2,j}(y)=(y-y_j)(y-y_{j+1})$. The desired result (3.19) follows from $\|w\|_{1,S_\Gamma}\leq C\overline{\|w\|}_{1,S_\Gamma}$ for $w\in V_h^0$. This completes the proof of Lemma 3.3. ■

Theorem 3.1. *Let $u\in C^3(\bar{S}_0)$ and $f\in C^2(\bar{S}_0)$, and* **A1–A3** *hold. Then for the FDM solution u_h^D with nine nodes, we have*

$$\overline{\|u-u_h^D\|}_1\leq C\left\{h^2\|u\|_{3,\infty,S_0}+h^2\|f\|_{2,\infty,S_0}+\overline{\||f\||}\right\}, \tag{3.27}$$

where $\overline{\||f\||}$ is given in Lemma 3.3, *and $\|u\|_{n,\infty,S_0}=\max_{\substack{i+j=k\leq n\\(x,y)\in S_0}}\partial^k u/\partial x^i\partial y^{k-i}$. Moreover, putting $r=(p+1)\mu$, we have*

$$\overline{\|u-u_h^D\|}_1=\begin{cases}O(h^r), & \text{if } r<2,\\ O\left(h^2\sqrt{\ln\frac{1}{h}}\right)=O(h^{2-\delta}),\quad 0<\delta<<1, & \text{if } r=2,\\ O(h^2), & \text{if } r>2.\end{cases} \tag{3.28}$$

Proof. The first bound in Eq. (3.27) is given directly from Li et al. (2003), Eq. (3.1), and Lemmas 3.1 and 3.3. Next from **A1–A3** we have for $i > 1$,

$$\begin{aligned}&\iint_{\square_{ij}} \kappa_{1,i}^2(x)u_{xxx}^2(x, y_j) = O(h_i^5 x_i^{2\sigma-6} k_j),\\ &\iint_{\square_{ij}} f_x^2(x-x_i)^2(x-x_{i+1})^2 = O(h_i^5 x_i^{2\nu-2} k_j),\\ &\|f_{xx}(x-x_i)(x-x_{i+1})\|_{-1,\square_{ij}}^2 = O(h_i^5 x_i^{2\nu-2} k_j).\end{aligned} \tag{3.29}$$

Since $h_i = x_i - x_{i-1} = O(h(ih)^p)$ and $x_i = O((ih)^{p+1})$ from the stretching function in Eq. (2.21), we have the following bounds for $i > 1$, by following the arguments in Yamamoto (2002),

$$\begin{aligned}&\iint_{\square_{ij}} \kappa_{1,i}^2(x)u_{xxx}^2(x, y_j) = O(i^{(2\sigma-1)(p+1)-5} \times h^{(2\sigma-1)(p+1)} k_j),\\ &\iint_{\square_{ij}} f_x^2(x-x_i)^2(x-x_{i+1})^2 = O(i^{(2\nu+3)(p+1)-5} \times h^{(2\nu+3)(p+1)} k_j).\end{aligned} \tag{3.30}$$

For $i = 0$, the integration by manipulation gives

$$\begin{aligned}&\iint_{\square_{0j}} \kappa_{1,i}^2(x)u_{xxx}^2(x, y_j) \le C\int_0^{k_j} dy \int_0^{h_1} x^4 x^{2\sigma-6}\, dx\\ &\quad = Ck_j \frac{h_1^{2\sigma-1}}{(2\sigma-1)} = Ck_j \frac{h^{(2\sigma-1)(p+1)}}{(2\sigma-1)},\\ &\iint_{\square_{0j}} f_x^2 x^2 (x-x_1)^2 \le Ck_j \frac{h^{(2\sigma-1)(p+1)}}{(2\sigma-1)}.\end{aligned} \tag{3.31}$$

Other bounds can be obtained similarly. Since $\sigma = 1/2 + \mu$ and $\nu = (-3/2) + \mu$, the last term on the right hand side in Eq. (3.27) has the bound,

$$\overline{|||f|||} = \sqrt{\sum_{\square_{ij}\in S_\Gamma} \left\{ \iint_{\square_{ij}} p_{ij}^2 + \|q_{ij}\|_{-1,\square_{ij}}^2 \right\}} \le C\sum_i \left\{ i^{2\mu(p+1)-5} \times h^{2\mu(p+1)} \right\}^{1/2}, \tag{3.32}$$

where C is a constant independent of h. Therefore, when $(p+1)\mu = r < 2$ we have $O(h^r)$. Also when $(p+1)\mu = 2$, we have $O(h^2\sqrt{\ln(1/h)})$ by noting $\sum_i(1/i) = O(\ln 1/h)$ from $\max i = O(1/h)$, see Yamamoto (2002). Moreover when $(p+1)\mu > 2$, we have $O(h^2)$ from Eqs. (3.27) and (3.32). This completes the proof of Theorem 3.1. ■

4. ANALYSIS FOR u_h WITH APPROXIMATION OF DIVERGENCE INTEGRATION

Since the bounds of first and third terms on the right hand side of Eq. (2.25) have been provided in Sec. 3 already, we focus only on the error estimates for the second term involving approximation of divergence integration. To this end, we put two additional assumptions.

A4. Suppose that the solution u is smooth enough except the derivatives with respect to x at $x=0$, which are unbounded and satisfy

$$\sup_{x\in(0,1)} x^{i-\sigma}\left|\frac{\partial^i}{\partial x^i}u(x,y)\right| \le C, \quad i=0,1,2,\dots, \tag{4.1}$$
$$\sup_{x\in(0,1)} x^{i-\nu}\left|\frac{\partial^i}{\partial x^i}f(x,y)\right| \le C, \quad i=0,1,2,$$

where $\sigma = 1/2+\mu, \nu=-3/2+\mu$, and $0<\mu<3/2$.

A5. Let $S_0 = S\backslash S_\Gamma$ and $S_\Gamma = \{(x,y)|0<x<a<1, 0<y<1\}$. Along the x direction, the partition is quasiuniform except h_i near $x=0$, where x_i in S_Γ are chosen as in **A3**. Denote $h=\max_{ij}\{h_i,k_j\}$. Hence, there exist the bounds,

$$\frac{h}{\min_j k_j} \le C, \quad \frac{h}{\min_{(i,j)\in S_0} h_i} \le C. \tag{4.2}$$

Since solution u is smooth along the y axis, we may choose the almost uniform partition of y_j such that $|k_{j+1}-k_j| = O(h^2)$. The analysis for the refinements of y_j also as A3 will be reported elsewhere.

Lemma 4.1. *Let* **A5** *hold. Then we have*

$$\left|\left(\widehat{\iint_S} - \iint_S\right)(u_I)_x v_x + (u_I)_y v_y\right|$$
$$\le C\left\{h^2\|\tilde u_{xyy}\|_{0,S} + h^3\|\tilde u_{xyyy}\|_{0,S} + T\right\} \times \overline{\|v\|}_1, \quad v\in V_h^0, \tag{4.3}$$

where

$$T = \sqrt{\sum_{ij}\|h_i^3\tilde u_{xxxx}\|^2_{0,\square_{ij}}} + h\sqrt{\sum_{ij}\|h_i^2\tilde u_{xxxy}\|^2_{0,\square_{ij}}}$$
$$+ h^2\sqrt{\sum_{ij}\|h_i\tilde u_{xxyy}\|^2_{0,\square_{ij}}} + h^2\sqrt{\sum_{ij}\left\|\frac{k_j^2}{h_i}\tilde u_{yyyy}\right\|^2_{0,\square_{ij}}}, \tag{4.4}$$

$\tilde u = u(\xi,\eta)$ *and* $(\xi,\eta)\in\square_{ij}$.

Proof. We have

$$\left(\widehat{\iint_S} - \iint_S\right)(u_I)_x v_x = \sum_{ij}\left(\widehat{\iint_{\square_{ij}}} - \iint_{\square_{ij}}\right)(u_I)_x v_x. \tag{4.5}$$

After some manipulation, we obtain

$$\iint_{\square_{ij}} (u_I)_x v_x = \frac{k_j}{3h_i}\{(u_4 - u_3)(v_4 - v_3) + (u_2 - u_1)(v_2 - v_1) + \frac{1}{2}(u_4 - u_3)(v_2 - v_1) + \frac{1}{2}(u_2 - u_1)(v_4 - v_3)\}, \quad (4.6)$$

and

$$\widehat{\iint}_{\square_{ij}} (u_I)_x v_x = \frac{k_j}{2h_i}\{(u_4 - u_3)(v_4 - v_3) + (u_2 - u_1)(v_2 - v_1)\},$$

where points 1,2,3,4 denote $(i,j), (i+1,j), (i,j+1), (i+1,j+1)$ respectively. Hence we have

$$\left(\widehat{\iint}_{\square_{ij}} - \iint_{\square_{ij}}\right)(u_I)_x v_x = \frac{k_j}{6h_i}(u_1 + u_4 - u_2 - u_3)(v_1 + v_4 - v_2 - v_3). \quad (4.7)$$

From the Taylor formulas, the solutions can be expanded at the center O at $(i + (1/2), j + (1/2))$,

$$u(i+1, j+1) = u_o + \left(\frac{h_i}{2}\frac{\partial}{\partial x} + \frac{k_j}{2}\frac{\partial}{\partial y}\right)u_o + \frac{1}{2}\left(\frac{h_i}{2}\frac{\partial}{\partial x} + \frac{k_j}{2}\frac{\partial}{\partial y}\right)^2 u_o + \frac{1}{3!}\left(\frac{h_i}{2}\frac{\partial}{\partial x} + \frac{k_j}{2}\frac{\partial}{\partial y}\right)^3 u_o + \frac{1}{4!}\left(\frac{h_i}{2}\frac{\partial}{\partial x} + \frac{k_j}{2}\frac{\partial}{\partial y}\right)^4 \tilde{u}, \quad (4.8)$$

where $\tilde{u} = u(\xi, \eta), (\xi, \eta) \in \square_{ij}$, and (ξ, η) may be different in different places. We then obtain after some manipulation,

$$u_1 + u_4 - u_2 - u_3 = h_i k_j (u_{xy})_o + \frac{k_j^4}{4 \cdot 4!}\tilde{u}_{yyyy} + b_1 h_i k_j^3 \tilde{u}_{xyyy} + b_2 h_i^2 k_j^2 \tilde{u}_{xxyy} + b_3 h_i^3 k_j \tilde{u}_{xxxy} + b_4 h_i^4 \tilde{u}_{xxxx}, \quad (4.9)$$

where $u_{x^i y^j} = (\partial^{i+j} u / \partial x^i \partial y^j)$, and b_ℓ are constants independent of h_i and k_j. Then we have from Eq. (4.7)

$$\left(\widehat{\iint}_S - \iint_S\right)(u_I)_x v_x = \sum_{ij}\left(\widehat{\iint}_{\square_{ij}} - \iint_{\square_{ij}}\right)(u_I)_x v_x = \frac{k_j}{6h_i}\left\{h_i k_j (u_{xy})_o + \frac{k_j^4}{4 \cdot 4!}\tilde{u}_{yyyy} + \sum_{\ell=1}^{4} b_\ell h_i^\ell k_j^{4-\ell} \tilde{u}_{x^\ell y^{4-\ell}}\right\} \times (v_1 + v_4 - v_2 - v_3). \quad (4.10)$$

Below, we will derive bounds of all terms on the right hand side of Eq. (4.10). Let the difference grids along y axis be $y_j, j = 0, 1, \ldots, N$, and denote $k_j = y_j - y_{j-1}$.

Since $v_{i,N} = v_{i,0} = 0$ from the Dirichlet condition on Γ, we obtain

$$\begin{aligned}
&\sum_{ij} k_j^2 (u_{xy})_o (v_1 + v_4 - v_2 - v_3) \\
&\quad = \sum_i \sum_{j=1}^{N-1} \{k_{j+1}^2 (u_{xy})_{i+(1/2),j+(1/2)} - k_j^2 (u_{xy})_{i+(1/2),j-(1/2)}\}(v_{i+1,j} - v_{ij}) \\
&\quad = \sum_i \sum_{j=1}^{N-1} k_j^2 \{(u_{xy})_{i+(1/2),j+(1/2)} - (u_{xy})_{i+(1/2),j-(1/2)}\}(v_{i+1,j} - v_{ij}) \\
&\qquad + \sum_i \sum_{j=1}^{N-1} \{(k_{j+1}^2 - k_j^2)(u_{xy})_{i+(1/2),j+(1/2)}\}(v_{i+1,j} - v_{ij}) \\
&\quad = T_A + T_B.
\end{aligned} \tag{4.11}$$

For the first term on the right hand side of the above equation, we have

$$\begin{aligned}
|T_A| &= \left| \sum_i \sum_{j=1}^{N-1} k_j^2 \{(u_{xy})_{i+(1/2),j+(1/2)} - (u_{xy})_{i+(1/2),j-(1/2)}\}(v_{i+1,j} - v_i) \right| \\
&= \left| \sum_i \sum_{j=1}^{N-1} k_j \frac{k_{j+1} + k_j}{2} (h_i h_j)(\tilde{u}_{xyy})_{i+(1/2),j} \frac{v_{i+1,j} - v_{ij}}{h_i} \right| \\
&\leq Ch^2 \|\tilde{u}_{xyy}\|_{0,S} \overline{\|v\|}_1,
\end{aligned} \tag{4.12}$$

in the last step of the above equation, we have also used the Schwarz inequality. Next, since $|k_{j+1} - k_j| = O(h^2)$ and $(k_{j+1} + k_j)/k_j \leq C$, we have

$$\begin{aligned}
|T_B| &= \left| \sum_i \sum_{j=1}^{N-1} \{(k_{j+1}^2 - k_j^2)(u_{xy})_{i+(1/2),j+(1/2)}\}(v_{i+1,j} - v_{ij}) \right| \\
&= \left| \sum_i \sum_{j=1}^{N-1} (k_{j+1} - k_j) \left(\frac{k_{j+1} + k_j}{k_j} \right) (h_i k_j)(u_{xy})_{i+(1/2),j+(1/2)} \frac{v_{i+1,j} - v_{ij}}{h_i} \right| \\
&\leq Ch^2 \|\tilde{u}_{xy}\|_{0,S} \overline{\|v\|}_1.
\end{aligned} \tag{4.13}$$

Combining Eqs. (4.11), (4.12), and (4.13) gives

$$\left| \sum_{ij} k_j^2 (u_{xy})_o (v_1 + v_4 - v_2 - v_3) \right| \leq Ch^2 \{\|\tilde{u}_{xy}\|_{0,S} + \|\tilde{u}_{xyy}\|_{0,S}\} \overline{\|v\|}_1. \tag{4.14}$$

This is the bound of the first term on the right hand side of Eq. (4.10).

Below we will estimate the bounds of two terms on the right hand side of Eq. (4.10). We have

$$\begin{aligned}
&\left|\sum_{ij}\frac{k_j}{h_i}k_j^4(\tilde{u}_{yyyy})(v_1+v_4-v_2-v_3)\right| \\
&\quad = \left|\sum_{ij}(h_ik_j)\frac{k_j^4}{h_i}(\tilde{u}_{yyyy})(\frac{v_4-v_3}{h_i}-\frac{v_2-v_1}{h_i})\right| \qquad (4.15)\\
&\quad \le Ch^2\sqrt{\sum_{ij}\left\|\frac{k_j^2}{h_i}\tilde{u}_{yyyy}\right\|^2_{0,\square_{ij}}}\cdot\overline{\|v\|}_1.
\end{aligned}$$

Next, we obtain

$$\begin{aligned}
&\left|\sum_{ij}\frac{k_j}{h_i}h_i^3k_j(\tilde{u}_{xxxy})(v_1+v_4-v_2-v_3)\right| \\
&\quad \le \sum_{ij}h_i^3k_j^2|\tilde{u}_{xxxy}|\cdot\left(\left|\frac{v_4-v_3}{h_i}\right|+\left|\frac{v_2-v_1}{h_i}\right|\right) \qquad (4.16)\\
&\quad \le Ch\sqrt{\sum_{ij}\|h_i^2\tilde{u}_{xxxy}\|^2_{0,\square_{ij}}}\cdot\overline{\|v\|}_1.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left|\sum_{ij}\frac{k_j}{h_i}h_i^4(\tilde{u}_{xxxx})(v_1+v_4-v_2-v_3)\right| \le C\sqrt{\sum_{ij}\|h_i^3\tilde{u}_{xxxx}\|^2_{0,\square_{ij}}}\cdot\overline{\|v\|}_1,\\
&\left|\sum_{ij}\frac{k_j}{h_i}h_i^2k_j^2(\tilde{u}_{xxyy})(v_1+v_4-v_2-v_3)\right| \le Ch^2\sqrt{\sum_{ij}\|h_i\tilde{u}_{xxyy}\|^2_{0,\square_{ij}}}\cdot\overline{\|v\|}_1,\\
&\left|\sum_{ij}\frac{k_j}{h_i}h_ik_j^3(\tilde{u}_{xyyy})(v_1+v_4-v_2-v_3)\right| \le Ch^3\sqrt{\sum_{ij}\|\tilde{u}_{xyyy}\|^2_{0,\square_{ij}}}\cdot\overline{\|v\|}_1\\
&\quad = Ch^3\|\tilde{u}_{xyyy}\|_{0,S}\cdot\overline{\|v\|}_1. \qquad (4.17)
\end{aligned}$$

Combining Eqs. (4.5), (4.10), (4.14)–(4.17) yields the bound,

$$\begin{aligned}
&\left|\left(\widehat{\iint_S}-\iint_S\right)(u_I)_xv_x\right| \\
&\quad \le C\left\{h^2(\|\tilde{u}_{xy}\|_{0,S}+\|\tilde{u}_{xyy}\|_{0,S})+h^3\|\tilde{u}_{xyyy}\|_{0,S}+T\right\}\times\overline{\|v\|}_1,
\end{aligned}$$

where T is given in Eq. (4.4).

Moreover, we can obtain similarly,

$$\left(\widehat{\iint}_{\square_{ij}} - \iint_{\square_{ij}}\right)(u_I)_y v_y = \frac{h_i}{6k_j}(u_1 + u_4 - u_2 - u_3)(v_1 + v_4 - v_2 - v_3). \tag{4.18}$$

Under **A5**, we have

$$\begin{aligned}\left|\left(\widehat{\iint}_{\square_{ij}} - \iint_{\square_{ij}}\right)(u_I)_y v_y\right| &= \frac{h_i^2}{k_j^2}\left|\left(\widehat{\iint}_{\square_{ij}} - \iint_{\square_{ij}}\right)(u_I)_x v_x\right| \\ &\leq C\left|\left(\widehat{\iint}_{\square_{ij}} - \iint_{\square_{ij}}\right)(u_I)_x v_x\right|,\end{aligned} \tag{4.19}$$

and then

$$\left|\left(\widehat{\iint}_{S} - \iint_{S}\right)(u_I)_y v_y\right| \leq C\left|\left(\widehat{\iint}_{S} - \iint_{S}\right)(u_I)_x v_x\right|. \tag{4.20}$$

The desired result in Eq. (4.3) follows from Eqs. (4.18) and (4.20). This completes the proof of Lemma 4.1. ■

Based on Lemma 4.1, we have the following theorem.

Theorem 4.1. *Let* **A4–A5** *hold, where the solution u is continuously differentiable of order four, expect with respect to x at $x = 0$. Then when $\sigma > (1/2) + (1/2)$ (e.g., $\mu > 1/2$), and choose $(p+1)\mu > 2$, there exists the best order of bounds,*

$$\sup_{v \in V_h^0} \frac{1}{\|v\|_1}\left|\left(\widehat{\iint}_{S} - \iint_{S}\right)\nabla u_I \nabla v\right| = O(h^2). \tag{4.21}$$

Moreover, for $\sigma = 1/2 + \mu$, putting $r = (p+1)\mu$, we have

$$\|u - u_h\|_1 = \begin{cases} O(h^r), & \text{if } r < 2, \\ O(h^2\sqrt{\ln \frac{1}{h}}) = O(h^{2-\delta}), \quad 0 < \delta << 1, & \text{if } r = 2, \\ O(h^2), & \text{if } r > 2. \end{cases} \tag{4.22}$$

Proof. We will derive more explicit bounds for all terms on the right hand side of Eq. (4.3) in Lemma 4.1. Since $\|u_{xy^\ell}\|_{0,S}$ are bounded from $\sigma > 1/2$, there exists the bound,

$$h^2(\|\tilde{u}_{xy}\|_{0,S} + \|\tilde{u}_{xyy}\|_{0,S}) + h^3\|\tilde{u}_{xyyy}\|_{0,S} \leq Ch^2. \tag{4.23}$$

By noting $h_i = O(h(ih)^p)$ and $x_i = O((ih)^{p+1})$, we have

$$\begin{aligned}
\sum_{ij} \|h_i^3 \tilde{u}_{xxxx}\|^2_{0,\square_{ij}} &= \sum_{ij} \iint_{\square_{ij}} h_i^6 \tilde{u}^2_{xxxx} \\
&\le C \sum_{ij} h_i^7 x_i^{2\sigma-8} k_j = C \sum_{ij} i^{(2\sigma-1)(p+1)-7} \cdot h^{(2\sigma-1)(p+1)} k_j, \\
\sum_{ij} h^2 \|h_i^2 \tilde{u}_{xxxy}\|^2_{0,\square_{ij}} &= \sum_{ij} h^2 \iint_{\square_{ij}} h_i^4 \tilde{u}^2_{xxxy} \le C \sum_{ij} h^2 \cdot h_i^5 x_i^{2\sigma-6} k_j \\
&= C \sum_{ij} h^2 \cdot (i^{(2\sigma-1)(p+1)-5} \cdot h^{(2\sigma-1)(p+1)} k_j) \\
&= C \sum_{ij} (ih)^2 \cdot (i^{(2\sigma-1)(p+1)-7} \cdot h^{(2\sigma-1)(p+1)} k_j), \\
\sum_{ij} h^4 \|h_i \tilde{u}_{xxyy}\|^2_{0,\square_{ij}} &\le C \sum_{ij} (ih)^4 \cdot (i^{(2\sigma-1)(p+1)-7} \cdot h^{(2\sigma-1)(p+1)} k_j).
\end{aligned} \tag{4.24}$$

More attention is paid to the bound of the last term, $h^2 \sqrt{\sum_{ij} \| \frac{k_j^2}{h_i} \tilde{u}_{yyyy} \|^2_{0,\square_{ij}}}$, on the right hand side of Eq. (4.4). Since $u_{yyyyy} \in C(S)$, $k_j = O(h)$, and $h_i = O(i^p h^{p+1})$, we have

$$\begin{aligned}
h^4 \sum_{ij} \left\| \frac{k_j^2}{h_i} \tilde{u}_{yyyy} \right\|^2_{0,\square_{ij}} &= h^4 \sum_{ij} \iint_{\square_{ij}} \frac{k_j^4}{h_i^2} \tilde{u}^2_{yyyyy} \\
&\le Ch^4 \sum_{ij} \frac{k_j^4}{h_i} k_j \le Ch^4 \left(\sum_i \frac{h^{4-(p+1)}}{i^p} \right) \left(\sum_j k_j \right) \le Ch^4,
\end{aligned} \tag{4.25}$$

provided that $p+1 < 4$. Hence, when $\sigma > 1/2 + 1/2 = 1$(i.e., $\mu > 1/2$), we obtain $p+1 < 4$ from $p+1 = 2/\mu$, and then the best order $O(h^2)$ in Eq. (4.21) from Lemma 4.1, Eqs. (4.23)–(4.25). Other results of Eq. (4.22) can be derived from Eq. (2.25), Lemma 4.1, and Theorem 3.1 similarly. This completes the proof of Theorem 4.1. ■

Compared with the results given in Theorem 3.1, there is a limitation of $\mu > 1/2$ in Theorem 4.1, which may be relaxed from Theorem 4.2 given below. From the viewpoint of stability, we cannot choose the case $\mu \to 0$. Since the condition number of the associated matrix **A** resulting from Eq. (2.4) is given in Strang and Fix (1973) by

$$\mathrm{Con}(\mathbf{A}) = \frac{\lambda_{\max}(\mathbf{A})}{\lambda_{\min}(\mathbf{A})} = O\left(\frac{1}{h^2_{\min}}\right) = O\left(\frac{1}{h_1^2}\right) = O\left(\frac{1}{h^{2(p+1)}}\right), \tag{4.26}$$

where $\lambda_{\max}(\mathbf{A})$ and $\lambda_{\min}(\mathbf{A})$ are the maximal and minimal eigenvalues of matrix **A**, respectively. When μ is very small, the optimal $p+1 > 2/\mu$ (or $p+1 = 2/\mu$) is large and Con(**A**) is extremely huge to yield poor solutions. For the numerical computation in Sec. 5, $p+1 = p_1 < 4$, which is guaranteed by Theorem 4.1.

Theorem 4.2. *Let* **A4–A5** *hold, where the solution u is continuously differentiable of order* $2n(n \geq 2)$, *except with respect to x at* $x = 0$. *Then when* $\sigma > 1/2 + 1/2(n-1)$ (*i.e.*, $\mu > 1/2(n-1)$), *and choose* $(p+1)\mu > 2$, *there exist the best order* $O(h^2)$ *in Eq.* (4.21) *and the superconvergence Eq.* (4.22) *for the Shortley–Weller approximation solution.*

Proof. When $n = 2$, Theorem 4.2 leads to Theorem 4.1. Hence we only give a proof for the case of $n = 3$. By the Taylor formulas, we obtain

$$\begin{aligned} u_1 + u_4 - u_2 - u_3 = h_i k_j (u_{xy})_o &+ \frac{1}{4!} h_i^3 k_j (u_{xxxy})_o + \frac{1}{4!} h_i k_j^3 (u_{xyyy})_o \\ &+ \frac{4}{6!}\left(\frac{k_j}{2}\right)^6 \tilde{u}_{yyyyyy} + \sum_{\ell=1}^{6} b_\ell^* h_i^\ell k_j^{6-\ell} \tilde{u}_{x^\ell y^{6-\ell}}, \end{aligned} \tag{4.27}$$

where b_ℓ^* are constants independent of h_i and k_j. By following the arguments in Theorem 4.1,

$$\begin{aligned} &\left| \left(\widehat{\iint_S} - \iint_S \right) \nabla u_I \nabla v \right| \\ &\leq C \Bigg\{ h^2 \|\tilde{u}_{xyy}\|_{0,S} + h \sqrt{\sum_{ij} \|h_i^2 \tilde{u}_{xxxy}\|^2_{0,\square_{ij}}} + h^3 \|\tilde{u}_{xyyy}\|_{0,S} \\ &\quad + h^2 \sqrt{\sum_{ij} \left\| \frac{k_j^4}{h_i} \tilde{u}_{yyyyyy} \right\|^2_{0,\square_{ij}}} + \sqrt{\sum_{\ell=2}^{6} h^{6-\ell} \sum_{ij} \|h_i^{\ell-1} \tilde{u}_{x^\ell y^{6-\ell}}\|^2_{0,\square_{ij}}} \Bigg\} \cdot \overline{\|v\|}_1. \end{aligned} \tag{4.28}$$

The following term is $O(h^2)$:

$$\begin{aligned} h^2 \sqrt{\sum_{ij} \left\| \frac{k_j^4}{h_i} \tilde{u}_{yyyyyy} \right\|^2_{0,\square_{ij}}} &\leq Ch^2 \sqrt{\sum_{ij} \frac{k_j^8}{h_i} k_j} \\ &\leq Ch^2 \sqrt{\left(\sum_i \frac{h^{8-(p+1)}}{i^p} \right) \left(\sum_j k_j \right)} \leq Ch^2, \end{aligned} \tag{4.29}$$

provided that $p + 1 < 8$ which implies $\mu > 2/p + 1 = 2/8 = 1/2(n-1)$ for $n = 3$. The analysis for other terms in Eq. (4.28) is similar. This completes the proof of Theorem 4.2. ■

Based on Theorems 3.1, 4.1, and 4.2, the best superconvergence $O(h^2)$ of derivatives can also be achieved for the Shortley–Waller approximation.

5. NUMERICAL EXPERIMENTS

Choose Eqs. (2.1) and (2.2), $S = \{(x, y)|0 < x < 1, 0 < y < 1\}$ and $u = x^\sigma(x-1)\times \sin \pi y$. Then $g \equiv 0$ on ∂S, and f is given by the computed function from Eq. (2.1) with $f = [\sigma(\sigma-1)x^{\sigma-2} - (\sigma+1)\sigma x^{\sigma-1} - \pi^2 x^\sigma + \pi^2 x^{\sigma+1}] \times \sin \pi y$ in S. For the sake of simplicity, let $[0, 1]$ be split into $[0, 1/4]$ and $[1/4, 1]$, and the uniform division be chosen on $[1/4, 1]$: $x_i = 1/4 + ih, i = 0, 1, 2, ..., N_b$, where $h = (3/4)(1/N_b)$. However, the local refinement of difference grids is chosen on $[0, 1/4]$ by

$$x_i = \frac{1}{4}\psi\left(\frac{i}{N_a}\right), \quad i = 0, 1, \ldots, N_a, \tag{5.1}$$

where $\psi(x) = (1-x)^{p_1}$ and $p_1 = p + 1$. Since the solution $u(x, y)$ is smooth along y, we choose the uniform grids along y: $y_j = jh_1$, where $h_1 = 1/N$. The interior difference grids $(i, j) = (x_i, y_j)$, $1 \le i \le N_x - 1$, $1 \le j \le N - 1$, where $N_x = N_a + N_b$. Note that the solution and partition of x_i and y_j satisfy all the conditions of Theorems 3.1, 4.1, and 4.2. Hence the superconvergence $O(h^2)$ of the solution derivatives can be achieved.

We first choose $\sigma = 7/6$; numerical results are provided in Tables 1–4 with $p_1 = 1, 2, 3, 3.5$. All numerical computation in Tables 1–6 is carried out in double precision. Since $p_1(\sigma - 1/2) = (2/3)p_1 = r$, we expect from Theorems 3.1 and 4.1,

$$\overline{\|\epsilon\|}_1 = O(h^{2/3}) \quad \text{for } p_1 = 1, \tag{5.2}$$

$$\overline{\|\epsilon\|}_1 = O(h^{4/3}) \quad \text{for } p_1 = 2, \tag{5.3}$$

$$\overline{\|\epsilon\|}_1 = O(h^2\sqrt{\ln\frac{1}{h}}), \quad 0 < \delta << 1, \quad \text{for } p_1 = 3, \tag{5.4}$$

$$\overline{\|\epsilon\|}_1 = O(h^2), \quad \text{for } p_1 = 3.5, \tag{5.5}$$

Table 1. Errors and condition numbers with $\sigma = 7/6$ and $p_1 = 1$.

N, N_a, N_b	4,5,3	8,10,6	16,20,12	32,40,24
$\overline{\|\epsilon\|}_0$	0.270(−2)	0.747(−3)	0.224(−3)	0.759(−4)
$\overline{\|\epsilon\|}_1$	0.898(−2)	0.249(−2)	0.900(−3)	0.464(−3)
$\text{Max}_{ij}\|\epsilon_{ij}\|$	0.576(−2)	0.149(−2)	0.408(−3)	0.169(−3)
$\text{Max}_{ij}\|(\epsilon_x)_{i+1/2,j}\|$	0.198(−1)	0.842(−2)	0.761(−2)	0.705(−2)
$\text{Max}_{ij}\|(\epsilon_y)_{i,j+1/2}\|$	0.523(−2)	0.205(−2)	0.107(−2)	0.517(−3)
Av	0.236(−2)	0.655(−3)	0.215(−3)	0.789(−4)
Av_x	0.558(−2)	0.157(−2)	0.598(−3)	0.255(−3)
Av_y	0.144(−2)	0.472(−3)	0.269(−3)	0.150(−3)
ϵ_1	0.450(−2)	0.755(−3)	0.335(−3)	0.150(−3)
ϵ_2	0.567(−2)	0.114(−2)	0.363(−3)	0.705(−2)
$\text{Max}_j\|\epsilon_{1j}\|$	0.183(−2)	0.755(−3)	0.355(−3)	0.150(−3)
$\text{Max}_j\|(\epsilon_x)_{1/2,j}\|$	0.117(−1)	0.842(−2)	0.761(−2)	0.705(−2)
Con(**A**)	22.0	110	495	0.183(4)

Table 2. Errors and condition numbers with $\sigma = 7/6$ and $p_1 = 2$.

N, N_a, N_b	4,5,3	8,10,6	16,20,12	32,40,24
$\overline{\|\epsilon\|}_0$	0.248(−2)	0.620(−3)	0.155(−3)	0.386(−4)
$\overline{\|\epsilon\|}_1$	0.934(−2)	0.257(−2)	0.755(−3)	0.244(−3)
$\text{Max}_{ij}\|\epsilon_{ij}\|$	0.535(−2)	0.133(−2)	0.339(−3)	0.846(−4)
$\text{Max}_{ij}\|(\epsilon_x)_{i+1/2,j}\|$	0.341(−1)	0.309(−1)	0.253(−1)	0.202(−1)
$\text{Max}_{ij}\|(\epsilon_y)_{i,j+1/2}\|$	0.720(−2)	0.203(−2)	0.526(−3)	0.133(−3)
Av	0.144(−2)	0.310(−3)	0.711(−4)	0.170(−4)
Av_x	0.853(−2)	0.276(−2)	0.971(−3)	0.360(−3)
Av_y	0.197(−2)	0.463(−3)	0.113(−3)	0.280(−4)
ϵ_1	0.438(−2)	0.606(−3)	0.778(−4)	0.981(−5)
ϵ_2	0.535(−2)	0.108(−2)	0.151(−3)	0.194(−4)
$\text{Max}_j\|\epsilon_{1j}\|$	0.156(−3)	0.409(−4)	0.861(−5)	0.174(−5)
$\text{Max}_j\|(\epsilon_x)_{1/2,j}\|$	0.341(−1)	0.309(−1)	0.253(−1)	0.203(−1)
Con(**A**)	29.4	291	0.242(4)	0.195(5)

Table 3. Errors and condition numbers with $\sigma = 7/6$ and $p_1 = 3$.

N, N_a, N_b	4,5,3	8,10,6	16,20,12	32,40,24
$\overline{\|\epsilon\|}_0$	0.250(−2)	0.623(−3)	0.156(−3)	0.389(−4)
$\overline{\|\epsilon\|}_1$	0.106(−1)	0.287(−2)	0.767(−3)	0.203(−3)
$\text{Max}_{ij}\|\epsilon_{ij}\|$	0.539(−2)	0.134(−2)	0.341(−3)	0.851(−4)
$\text{Max}_{ij}\|(\epsilon_x)_{i+1/2,j}\|$	0.143	0.109	0.786(−1)	0.560(−1)
$\text{Max}_{ij}\|(\epsilon_y)_{i,j+1/2}\|$	0.697(−2)	0.196(−2)	0.509(−3)	0.129(−3)
Av	0.145(−2)	0.313(−3)	0.721(−4)	0.172(−4)
Av_x	0.228(−1)	0.816(−2)	0.296(−2)	0.107(−2)
Av_y	0.214(−2)	0.496(−2)	0.120(−3)	0.295(−4)
ϵ_1	0.440(−2)	0.608(−3)	0.780(−4)	0.983(−5)
ϵ_2	0.539(−2)	0.108(−2)	0.151(−3)	0.194(−4)
$\text{Max}_j\|\epsilon_{1j}\|$	0.259(−3)	0.247(−4)	0.224(−5)	0.200(−6)
$\text{Max}_j\|(\epsilon_x)_{1/2,j}\|$	0.143	0.109	0.786(−1)	0.560(−1)
Con(**A**)	93.4	0.197(4)	0.335(5)	0.548(6)

where $\epsilon = u - u_h$.

From Table 4 we can see asymptotic rates for $p_1 = 3.5$,

$$\overline{\|\epsilon\|}_0 = O(h^2), \overline{\|\epsilon\|}_1 = O(h^2), \text{Con}(\mathbf{A}) = O(h^{-4.5}), \tag{5.6}$$

$$Av = O(h^2), \quad Av_x = O(h^{1.5}), \quad Av_y = O(h^2), \tag{5.7}$$

$$\max_{ij}|\epsilon_{ij}| = O(h^2), \quad \max_{ij}|(\epsilon_x)_{i+1/2,j}| = O(h^{1/2}), \quad \max_{ij}|(\epsilon_y)_{i,j+1/2}| = O(h^2),$$

$$\epsilon_1 = O(h^3), \quad \epsilon_2 = O(h^3), \tag{5.8}$$

$$\max_j|\epsilon_{1j}| = O(h^4), \quad \max_j|(\epsilon_x)_{1/2,j}| = O(h^{1/2}), \tag{5.9}$$

Table 4. Errors and condition numbers with $\sigma = 7/6$ and $p_1 = 3.5$.

N, N_a, N_b	4,5,3	8,10,6	16,20,12	32,40,24
$\overline{\Vert\epsilon\Vert}_0$	0.252(−2)	0.630(−3)	0.157(−3)	0.393(−4)
$\overline{\Vert\epsilon\Vert}_1$	0.112(−1)	0.297(−2)	0.761(−3)	0.195(−3)
$\text{Max}_{ij}\vert\epsilon_{ij}\vert$	0.544(−2)	0.134(−2)	0.343(−3)	0.858(−4)
$\text{Max}_{ij}\vert(\epsilon_x)_{i+1/2,j}\vert$	0.228	0.163	0.111	0.748(−1)
$\text{Max}_{ij}\vert(\epsilon_y)_{i,j+1/2}\vert$	0.667(−2)	0.188(−2)	0.486(−3)	0.122(−3)
Av	0.153(−2)	0.326(−3)	0.745(−4)	0.178(−4)
Av_x	0.346(−1)	0.123(−1)	0.432(−2)	0.151(−2)
Av_y	0.215(−2)	0.494(−2)	0.119(−3)	0.293(−4)
ϵ_1	0.441(−2)	0.610(−3)	0.783(−4)	0.987(−5)
ϵ_2	0.543(−2)	0.108(−2)	0.152(−3)	0.195(−4)
$\text{Max}_j\vert\epsilon_{1j}\vert$	0.193(−3)	0.122(−4)	0.738(−6)	0.439(−7)
$\text{Max}_j\vert(\epsilon_x)_{1/2,j}\vert$	0.228	0.163	0.111	0.748(−1)
$\text{Con}(\mathbf{A})$	183	0.561(4)	0.136(6)	0.316(7)

where $\epsilon_x = u_x - (u_h)_x$ and $\epsilon_y = u_y - (u_h)_y$,

$$\begin{aligned} Av &= \frac{1}{\text{Num}}\sum_{ij}\left|\epsilon(i,j)\right|, \quad Av_x = \frac{1}{\text{Num}}\sum_{ij}\left|\epsilon_x(i+\frac{1}{2},j)\right|, \\ Av_y &= \frac{1}{\text{Num}}\sum_{ij}\left|\epsilon_y(i,j+\frac{1}{2})\right|, \\ \epsilon_1 &= \max_{i,j}\left\{|\epsilon_{1,j}|, |\epsilon_{2N-1,j}|, |\epsilon_{i,1}|, |\epsilon_{i,N-1}|\right\}, \\ \epsilon_2 &= \max_{i,j}\left\{|\epsilon_{2,j}|, |\epsilon_{2N-2,j}|, |\epsilon_{i,2}|, |\epsilon_{i,N-2}|\right\}, \end{aligned} \tag{5.10}$$

where Num is the total number of the errors related. The best superconvergence rate $\overline{\Vert\epsilon\Vert}_1 = O(h^2)$ is consistent with Eq. (5.5). Other convergence rates are in the range of $O(h^2)$ and $O(h^4)$, except the derivatives with respect to x have the reduced convergence rates: $Av_x = O(h^{1.5})$, $\max_{ij}|(\epsilon_x)_{i+1/2,j}| = O(h^{1/2})$ and $\max_j|(\epsilon_x)_{1/2,j}| = O(h^{1/2})$. Moreover, the condition numbers are large: $\text{Con}(\mathbf{A}) = O(h^{-4.5})$.

Looking at Tables 1–3, the ratios of $\overline{\Vert\epsilon\Vert}_1$ in the last two columns are given by

$$\begin{aligned} &\frac{0.900(-3)}{0.464(-3)} = 1.94 = 2^{0.95} = O(h^{8/9}), \quad p_1 = 1, \\ &\frac{0.755(-3)}{0.244(-3)} = 3.09 = 2^{1.62} = O(h^{3/2}), \quad p_1 = 2, \\ &\frac{0.767(-3)}{0.203(-3)} = 3.778 = 2^{1.92} = O(h^{2-\delta}), \quad 0 < \delta << 1, \quad p_1 = 3, \end{aligned} \tag{5.11}$$

which are all coincident with the predictions in Eqs. (5.2)–(5.4). When $p_1 = 1$, the poor rates, $O(h^{2/3})$ in theory and $O(h^{8/9})$ in computation, display a necessity using the local refinements to improve accuracy of the numerical solutions.

Next we choose $\sigma = 0.95$ and $p_1 = 4.4444444444$; numerical results are provided in Table 5, from which we can see

$$\overline{\|\epsilon\|}_0 = O(h^2), \overline{\|\epsilon\|}_1 = O(h^{2-\delta}), \quad \text{Con}(\mathbf{A}) = O(h^{-5.5}), \tag{5.12}$$

$$Av = O(h^2), \quad Av_x = O(h^{0.85}), \quad Av_y = O(h^2), \tag{5.13}$$

$$\max_{ij}|\epsilon_{ij}| = O(h^2), \quad \max_{ij}|(\epsilon_x)_{i+1/2,j}| = O(h^{-\delta}), \quad \max_{ij}|(\epsilon_y)_{i,j+1/2}| = O(h^2),$$

$$\epsilon_1 = O(h^3), \quad \epsilon_2 = O(h^3), \tag{5.14}$$

$$\max_{j}|\epsilon_{1j}| = O(h^3), \quad \max_{j}|(\epsilon_x)_{1/2,j}| = O(h^{-\delta}), \quad 0 < \delta << 1.$$

Although the derivatives $\max_{ij} |(\epsilon_x)_{i+1/2,j}| = O(h^{-\delta})$ with respect to x are divergent for $\sigma < 1$, the average nodal derivatives $Av_x = O(h^{0.85})$ are still convergent. Such results are also in good agreement with Theorems 3.1, 4.1, and 4.2.

Finally we choose $\sigma = 0.5$ and $p_1 = 4$ as in Fang (2001), Yamamoto (2002); numerical results are provided in Table 6, from which we can see

$$\overline{\|\epsilon\|}_0 = O(h^{1.75}), \overline{\|\epsilon\|}_1 = 1.10, \quad \text{Con}(\mathbf{A}) = O(h^{-5}), \tag{5.15}$$

$$Av = O(h^{1.75}), \quad Av_x = O(h^{-1}), \quad Av_y = O(h^{1.75}), \tag{5.16}$$

$$\max_{ij} |\epsilon_{ij}| = O(h^{1.75}), \quad \max_{ij} |(\epsilon_x)_{i+1/2,j}| = O(h^{-2}), \quad \max_{ij} |(\epsilon_y)_{i,j+1/2}| = O(h^{1.75}),$$

$$\epsilon_1 = O(h^2), \quad \epsilon_2 = O(h^2), \tag{5.17}$$

$$\max_{j} |\epsilon_{1j}| = O(h^2), \quad \max_{j} |(\epsilon_x)_{1/2,j}| = O(h^{-2}).$$

For $\sigma = 1/2$, the discrete errors in H^1 norms maintain invariant, $\overline{\|\epsilon\|}_1 = 1.10$, to verify the necessary condition (2.16), but the solution itself and the derivatives with respect to y are still convergent.

Table 5. Errors and condition numbers with $\sigma = 0.95$ and $p_1 = 4.4444444444$.

N, N_a, N_b	4,5,3	8,10,6	16,20,12	32,40,24
$\overline{\|\epsilon\|}_0$	0.370(−2)	0.925(−3)	0.231(−3)	0.578(−4)
$\overline{\|\epsilon\|}_1$	0.157(−1)	0.418(−2)	0.111(−2)	0.291(−3)
$\text{Max}_{ij}\|\epsilon_{ij}\|$	0.714(−2)	0.176(−2)	0.439(−3)	0.110(−3)
$\text{Max}_{ij}\|(\epsilon_x)_{i+1/2,j}\|$	0.698	0.802	0.932	1.09
$\text{Max}_{ij}\|(\epsilon_y)_{i,j+1/2}\|$	0.247(−2)	0.713(−3)	0.185(−3)	0.468(−4)
Av	0.272(−2)	0.570(−3)	0.130(−3)	0.311(−4)
Av_x	0.963(−1)	0.485(−1)	0.262(−1)	0.148(−1)
Av_y	0.913(−3)	0.227(−3)	0.556(−4)	0.138(−4)
ϵ_1	0.505(−2)	0.674(−3)	0.856(−4)	0.108(−4)
ϵ_2	0.714(−2)	0.125(−2)	0.168(−3)	0.215(−4)
$\text{Max}_j\|\epsilon_{1j}\|$	0.131(−3)	0.693(−5)	0.370(−6)	0.198(−7)
$\text{Max}_j\|(\epsilon_x)_{1/2,j}\|$	0.698	0.802	0.932	1.09
Con(**A**)	767	0.438(5)	0.208(7)	0.928(8)

Table 6. Errors and condition numbers with $\sigma = 0.5$ and $p_1 = 4$.

N, N_a, N_b	4,5,3	8,10,6	16,20,12	32,40,24
$\overline{\|\epsilon\|}_0$	0.169(−1)	0.514(−2)	0.152(−2)	0.439(−3)
$\overline{\|\epsilon\|}_1$	1.10	1.10	1.10	1.10
$\text{Max}_{ij}\|\epsilon_{ij}\|$	0.412(−1)	0.134(−1)	0.412(−2)	0.123(−2)
$\text{Max}_{ij}\|(\epsilon_x)_{i+1/2,j}\|$	76.6	306	0.123(4)	0.490(4)
$\text{Max}_{ij}\|(\epsilon_y)_{i,j+1/2}\|$	0.104	0.375(−1)	0.121(−1)	0.368(−2)
Av	0.230	0.626(−2)	0.178(−2)	0.509(−3)
Av_x	8.10	14.5	27.3	53.0
Av_y	0.458(−1)	0.148(−1)	0.462(−2)	0.140(−2)
ϵ_1	0.291(−1)	0.619(−2)	0.155(−2)	0.387(−3)
ϵ_2	0.412(−1)	0.945(−2)	0.228(−2)	0.560(−3)
$\text{Max}_j\|\epsilon_{1j}\|$	0.248(−1)	0.619(−2)	0.155(−2)	0.387(−3)
$\text{Max}_j\|(\epsilon_x)_{1/2,j}\|$	76.6	306	0.123(4)	0.490(4)
Con(**A**)	370	0.165(5)	0.572(6)	0.188(8)

6. CONCLUDING REMARKS

To close this article, let us make a few remarks.

1. The main results are given in Theorems 3.1, 4.1, and 4.2. Numerical experiments are provided to support the error analysis made. By a local refinement technique of the stretching function used in Yamamoto (2002), when $u \in C^{(1/2)+\mu}(\bar{S}_\Gamma)$ (e.g., $u \in O(x^{(1/2)+\mu}(1-x)^{(1/2)+\mu})$ and $u \in O(y^{(1/2)+\mu}(1-y)^{(1/2)+\mu})$ on the unit square S) and $f \in C^{-(3/2)+\mu}(\bar{S}_\Gamma)$, $0 < \mu < 3/2$, the superconvergence $O(h^2)$ of the solution derivatives can also be achieved.
2. When the divergence integration involves approximation, the analysis on superconvergence is more challenging. For the case $\sigma = 1/2 + \mu$ where $\mu \to 0$, the solution in **A4** is required infinitely continuously differentiable except with respect to x at $x = 0$, see Theorem 4.2. This is not realistic. However, when $\mu \to 0$, $p_1 = p + 1 \to \infty$ (also see Yamamoto (2002), the condition number Con(**A**) grows to infinity. In fact, the assumptions in Theorem 4.1 are reasonable and useful, i.e., the solutions are continuously differentiable of only order four, except with respect to x at $x = 0$. We obtain $p + 1 < 4$ which may satisfy the requirements of a large scale of application, see the numerical examples in Sec. 5.
3. The superconvergence analysis of derivatives in this article is valid to all three numerical methods: the Shortley–Weller approximation, the FDM with nine nodes and the bilinear FEM.
4. We note that the assumption $\sigma > 0$ in Fang (2001), Yamamoto (2002) is weaker than $\sigma > 1/2$ in A1. Let us consider an a posteriori computation for the nodal derivatives recovered by

$$(u_h)_x\left(i + \frac{1}{2}, j\right) = \frac{u_h(i+1, j) - u_h(i, j)}{h_i}, \tag{6.1}$$

where $u_h \in V_h$ is the solution of the FDM. Denote $\epsilon = u - u_h$. We obtain from Eq. (3.8)

$$\left|\epsilon_x\left(i+\frac{1}{2},j\right)\right| = \frac{1}{h_i}\left|\int_{x_i}^{x_{i+1}} \kappa_{1,i}(x)u_{xxx}(x,y_j)\,dx\right|, \tag{6.2}$$

where $\kappa_{1,i}(x)$ is given by Eq. (3.5). When $i=0$, we have from **A1–A3**,

$$\left|\epsilon_x\left(\frac{1}{2},j\right)\right| = \frac{1}{h_1}\left|\int_0^{x_1} \kappa_{1,i}(x)u_{xxx}(x,y_j)\,dx\right| = O(h^{(\sigma-1)(p+1)}) \to 0, \tag{6.3}$$

provided that $\sigma > 1$, which is even stronger than $\sigma > 1/2$ in Eq. (2.16) of **A1**.

5. This article is devoted mainly to the derivative error estimates for some engineering problems, e.g., fluid and mechanics problems, where the solution derivatives as velocity or stress are more important. When $\sigma \in (0, 1/2)$ and for the nodal solution errors only, reader may refer Fang (2001), Fang et al. (2001), Yamamoto (2001, 2002), and Yamamoto et al. (2001).
6. The assumption of the rectangular domain S is for the basic exploration on the analysis for unbounded derivatives near Γ of S. Here, let us briefly explain extensions of this article to non-rectangular domains; details are reported in a sequential article.

   In general, when S is a polygon, Eqs. (2.13) and (2.14) in **A1** should be replaced by

$$\sup_{d\in(0,1)} d^{i-\alpha_0}(1-d)^{i-\alpha_1}\left|\frac{\partial^i}{\partial n^i}u(x,y)\right| \le C_1, \quad i=1,2,3, \tag{6.4}$$

   where d is the distance between (x,y) and Γ, and n is the normal to $\partial S_\Gamma \cap \Gamma$. Since one system of difference grids is difficult to fix a polygon, in particular in the case involving local refinements near Γ, we may split S into several sub-polygons S_i (i.e., $S = \cup_i S_i$) by an interior boundary Γ_0. Different Cartesian coordinates are chosen in different subdomains S_i, and different systems of difference grids are used in S_i. We had better choose the same interior difference nodes just on Γ_0 so that the admissible piecewise bilinear and linear functions are continuous on the entire S. Then there exist no errors from Γ_0. Otherwise, coupling techniques can be employed along Γ_0, and an analysis should be made to evaluate errors from the coupling techniques, see Li (1998).

   For polygon S, local refinements for $\square_{ij}$ and triangles $\triangle_{ij}$ near Γ may also follow **A3**. By following this article and Li et al. (2003), only the superconvergence $O(h^{1.5})$ of solution derivatives can be obtained by the Shortley–Weller difference approximation for Poisson's equation.

   Other possible extensions of the analysis in this article are for a sectorial domain S, where the solution satisfies Eq. (6.4). In this case, the finite difference equations are formulated along the polar coordinates, and superconvergence of solution derivatives may also be carried out by following the lines of this article.

7. Although this article is confined mainly to a rectangular domain, there seems to be no complete convergence proof in the case of the use of the stretching function (2.21) for such singular problems even for a rectangular domain. The stretching functions are often used in the area of fluid dynamics. This article displays an interesting fact that even if the local truncation error diverges, the FDM solution may converge, also see Yamamoto (2002). Hence, the results in this article on discrete H^1 errors and those in Yamamoto (2002) on the maximal nodal errors for a rectangular domain yield a mathematical foundation for the stretching function in **A3** for an irregular domain, see Remark 6.

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REFERENCES

Atkinson, K. E. (1989). *An Introduction to Numerical Analysis*. 2nd Ed. New York: John Wiley & Sons.

Bramble, J. H., Hubbard, B. E. (1962). On the formulation of finite difference analogies of the Dirichlet problem for Poisson's equation. *Numer. Math.* 4:313–327.

Ciarlet, C. G. (1991). Basic error estimates for elliptic problems. In: Ciarlet, P. G., Lions, J. L., eds. North-Holland: Finite Element Methods (Part I).

Fang, Q. (2001). Convergence of finite difference methods for convection-diffusion problems with singular solutions. To appear in Proceeding of ICRACM 2001.

Fang, Q., Matsubara, T., Shogenji, Y., Yamamoto, T. (2001). Convergence of inconsistent finite difference scheme for Dirichlet problem whose solution has singular derivatives at the boundary. *Information* 4(2):161–170.

Li, Z. C. (1998). *Combined Methods for Elliptic Equations with Singularities, Interfaces and Infinities*. Dordrecht, Boston and London: Kluwer Academic Publishers.

Li, Z. C., Yamamoto, T., Fang, Q. (2003). Superconvergence of solution derivatives for the Shortley–Weller difference approximation of Poisson's equation, Part I. Smoothness problems. *J. Comp. and Appl. Math.* 152(2):307–333.

Liu, W. B., Tang, T. (2001). Error analysis for a Galerkin-spectral method with coordinate transformation for solving singularly perturbed problems. *Applied Numerical Mathematics* 38:315–345.

Strang, G., Fix, G. J. (1973). *An Analysis of the Finite Element Method*. North-Holland, Amsterdam: Prentice-Hall, Inc.

Tang, T., Trummer, M. R. (1996). Boundary layer resolving pseudospectral method for singular perturbation problems. *SIAM J. Scientific Computing* 17:430–428.

Yamamoto, T. (2001). Inversion formulas for tridiagonal matrices with applications to boundary value problems. *Numer. Funct. Anal. and Optimiz.* 22:357–385.

Yamamoto, T. (2002). Convergence of consistant and inconsistent finite difference schemes and an acceleration technique. *J. Comp. Appl. Math.* 140:849–866.

Yamamoto, T., Fang, Q., Chen, X. (2001). Superconvergence and nonsuperconvergence of the Shortley–Weller approximations for Dirichlet problems. *Numer. Funct. Anal. and Optimiz.* 22:455–470.

Yoshida, K., Tsuchiya, T. (2001). Recovered derivatives for the Shortley–Weller finite difference approximation. *Information* 4(3):267–277.