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Radial Basis Collocation Methods for Elliptic Boundary Value Problems

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Abstract—This paper introduces radial basis functions (RBF) into the collocation methods and the combined methods for elliptic boundary value problems First, the Ritz-Galerkin method (RGM) is chosen using the RBF, and the integration approximation leads to the collocation method of RBF for Poisson's equation Next, the combinations of RBF with finite-element method (FEM), finite-difference method (FDM), etc., can be easily formulated by following Li [1] and Hu and Li [2,3], but more analysis of inverse estimates is explored in this paper. Since the RBFs have the exponential convergence rates, and since the collocation nodes may be scattered in rather arbitrary fashions in various applications, the RBF may be competitive to orthogonal polynomials for smooth solutions Moreover, for singular solutions, we may use some singular functions and RBFs together. Numerical examples for smooth and singularity problems are provided to display effectiveness of the methods proposed and to support the analysis made. © 2005 Elsevier Ltd. All rights reserved.

Keywords—Radial basis functions, Collocation method, Combined method, Elliptic problems, Singularity problem, Motz's problem

1. INTRODUCTION

There have been many developments for radial basis functions (RBF) in recent years. The RBF can be used for the interpolatory tools for smooth solutions, $u \in C^\infty(S)$. The convergence of interpolants to a given continuous function has been discussed in the following works: Kansa [4]

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provided the surface approximations and partial derivative estimates, Madych [5] established several types of error bounds for multiquadric and related interpolators, Wu and Schaback [6] focused on local errors of scattered data interpolation by RBF in suitable variational formulation, and Yoon [7] regarded the convergence of RBF in an arbitrary Sobolev space. All of those reports show exponential convergence rates. Moreover, the applications of RBF have been given as follows. Kansa [4,8] presented a series of applications in computational fluid dynamics. Franke and Schaback [9] gave some theoretical foundational method for solving partial differential equations (PDEs). Wendland [10] derived error estimates for the solution of smooth problems. Cheng *et al.* [11] introduced the *h-c* meshless scheme for smooth problems, where numerical experiments were also provided. May-Duy and Tran-Cong [12,13] used the radial basis function network methods for Poisson's equations.

We may classify the collocation method as a special kind of spectral method, in which numerical solutions have high accuracy, but with high instability due to the large condition number. Fortunately, in practice, only a few terms of RBF are needed so that the condition number will not be very large, and useful numerical computation can be carried out even in double precision. Since by using MATHEMATICA, unlimited number of significant digits are available, the spectral methods using RBF are more promising.

In this paper, we consider the collocation method using RBF, simply called radial basis collocation method (RBCM). We derive inverse estimates and new error analysis. The collocation method is treated as Ritz-Galerkin method involving integration approximation. However, the integration quadrature is used in analysis only to satisfy the V_h-elliptic inequality, but not to reach the highly exponential convergence rates. More explanations are given in Section 3. The advantages of the radial basis collocation method are twofold.

(1) Source points of radial basis functions and collocation nodes may be scattered in rather arbitrary fashions in various applications, in which the solution domain is not confined in a rectangle. We need a dense set of collocation nodes in any irregular domain.
(2) Simplicity of the computed codes. A drawback of the radial basis collocation method is the high instability with large condition number.

This paper is organized as follows. In the next section, the radial basis functions are described, and in Section 3, the collocation methods for different boundary conditions and combinations of FEM are discussed. In Section 4, the inverse estimates are derived. In the last section, numerical experiments including smooth and singularity problems are carried out to display the effectiveness of the methods proposed, and to support the analysis made.

2. RADIAL BASIS FUNCTIONS

For surface fitting on scattered points, using the RBF shows remarkable advantages (see [4,6–9,14–17]). Based on the theory of finite-element method (FEM) in Ciarlet [18], the errors of numerical solutions for elliptic equations are, basically, those of optimal approximations of admissible functions to the true solutions. Hence, the RBF can be definitely applied to solving elliptic equations.

Let us describe the RBF. The multiqudrics, the thin-plate splines, and the Gaussian functions are defined by (see [11])

$$g_i(x) = \left(r_i^2 + c^2\right)^{n-3/2}, \qquad n = 1, 2, \dots, \tag{2.1}$$

$$g_i(x) = \begin{cases} r_i^{2n} \ln r_i, & n = 1, 2, \dots, \quad \text{in } R^2, \\ r_i^{2n-1}, & n = 1, 2, \dots, \quad \text{in } R^3, \end{cases} \tag{2.2}$$

$$g_i(x) = \exp\left(-\frac{r_i^2}{c^2}\right), \tag{2.3}$$

$$g_i(x) = \left(r_i^2 + c^2\right)^{n-3/2} \exp\left(-\frac{r_i^2}{a^2}\right), \qquad n = 1, 2, \dots, \tag{2.4}$$

where

$$x = (x, y)\,, \quad \text{in } R^2, \qquad \text{and} \qquad x = (x, y, z)\,, \quad \text{in } R^3.$$

The radius $r_i = \{(x - x_i)^2 + (y - y_i)^2\}^{1/2}$ in R^2 and $r_i = \{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2\}^{1/2}$ in R^3, where (x_i, y_i) and (x_i, y_i, z_i) are called source points of RBF. The constants c and a are shape parameters to be chosen later. When parameters c and a become large, the RBF become flat.

Choose a linear combination of RBF,

$$v = \sum_{i=1}^{n} a_i g_i(x)\,, \tag{2.5}$$

where the coefficients a_i are sought by the Ritz-Galerkin method or the collocation methods. Since v in (2.5) cannot equal to a constant exactly, we may add a constant into (2.5) (see [4,8]) to get

$$v = b_1 + \sum_{i=2}^{n} a_i \hat{g}_i(x)\,, \tag{2.6}$$

where

$$\hat{g}_i(x) = g_i(x) - g_1(x)\,. \tag{2.7}$$

Moreover, since v in (2.5) cannot equal to linear function exactly, either we may also add a linear function into (2.5) to get

$$v = b_1 + b_2 x + \sum_{i=3}^{n} a_i g_i^*(x)\,, \tag{2.8}$$

where

$$g_i^*(x) = g_i(x) - \frac{(x_2 - x_i)\, g_1(x) + (x_i - x_1)\, g_2(x)}{x_2 - x_1} \tag{2.9}$$

In general, we may add a polynomial of power m (see [6]) to get

$$v = \sum_{i=1}^{n} a_i g_i(x) + \sum_{i=1}^{m} b_i P_i(x)\,, \tag{2.10}$$

where $P_k(x) = \sum_{r+\ell\le k} x^r y^\ell$. In (2.10), $m = C_2^{k+1}$ in R^2 and $m = C_3^{k+2}$ in R^3, which denote the total number of power functions in polynomials with power $\le k$. For curve fitting, the coefficients are sought by satisfying

$$\sum_{i=1}^{n} a_i g_i(x_j) + \sum_{i=1}^{m} b_i P_i(x_j) = f_j, \qquad j = 1, 2, \dots, n, \tag{2.11}$$

and

$$\sum_{j=1}^{n} a_j P_i(x_j) = 0, \qquad i = 1, 2, \dots, m. \tag{2.12}$$

Based on the same ideas, we may add some singular functions $\psi_i(x)$ for fitting singular surfaces, or for solving singular problems of elliptic equations,

$$v = \sum_{i=1}^{n} a_i g_i(x) + \sum_{i=0}^{m} b_i \psi_i(x)\,, \tag{2.13}$$

where

$$\sum_{i=1}^{n} a_i g_i(x_j) + \sum_{i=0}^{m} b_i \psi_i(x_j) = f_j, \qquad j = 1, 2, \dots, n, \tag{2.14}$$

and

$$\sum_{j=1}^{n} a_j \psi_i(x_j) = 0, \qquad i = 0, 1, \dots, m. \tag{2.15}$$

Note that equation (2.15) may or may not be added into the computation.

For Motz's problem given in Section 5.2, a benchmark of singularity problems, we may add a few leading singular functions as

$$\psi_i(x) = r^{i+1/2} \cos\left(i + \frac{1}{2}\right)\theta, \qquad i = 0, 1, \dots, 5, \tag{2.16}$$

where (r, θ) are the polar coordinates with the origin (0,0), $r = \sqrt{x^2 + y^2}$, and $\tan\theta = y/x$. In Section 5, numerical experiments of Motz's problem are carried out to display the effectiveness of the RBCM.

3. DESCRIPTION OF RADIAL BASIS COLLOCATION METHOD

By following the approaches in [2,3] the RBF are chosen as the admissible functions in the Ritz-Galerkin methods. Since the RBFs do not satisfy the partial differential equation and the boundary conditions so that the residuals have to be enforced to zero at collocation points both in the solution domain and on its boundary. We obtain the collocation methods of RBF, simply denoted by RBCM, and combined methods of RBCM with other numerical methods. The optimal error bounds are provided, and the proofs are given in Section 3.1, or can be done by following [2,3] straightforward. The crucial inverse estimates for radial basis functions will be proven in the next section.

3.1. Radial Basis Collocation Method for Various Boundary Conditions

Consider the Poisson's equation with Robin boundary condition,

$$-\Delta u = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f(x, y), \qquad \text{in } S, \tag{3.1}$$

$$u_\nu|_{\Gamma_N} = q_1, \qquad \text{on } \Gamma_N, \tag{3.2}$$

$$(u_\nu + \beta u)|_{\Gamma_R} = q_2, \qquad \text{on } \Gamma_R, \tag{3.3}$$

where $\beta \geq \beta_0 > 0$, β_0 is constant, S is a polygon, $\partial S = \Gamma = \Gamma_N \cup \Gamma_R$, and u_ν is the outer normal derivative to ∂S. Assume $\mathrm{Meas}(\Gamma_R) > 0$, for guaranteeing the unique solution. We make two assumptions.

(A1) The solutions in S can be expanded as

$$v = \sum_{i=1}^{\infty} a_i g_i(x, y), \qquad \text{in } S, \tag{3.4}$$

where $g_i(x, y) \in C^2(\bar{S})$ are RBF, and a_i are the expansion coefficients.

(A2) The expansions in (3.4) converge exponentially to the true solutions u,

$$u = u_L + R_L, \tag{3.5}$$

where $u_L = \sum_{i=1}^{L} a_i g_i(x,y)$ and $R_L = \sum_{i=L+1}^{\infty} a_i g_i(x,y)$. Then,

$$\max_S |R_L| = O\left(\lambda^{c/\delta}\right), \tag{3.6}$$

where $c > 0$, $L > 1$, $0 < \lambda < 1$, and δ is the radial distance defined as

$$\delta = \sup_{(x,y)\in S} \left\{ \inf_i \left\{ (x-x_i)^2 + (y-y_i)^2 \right\}^{1/2} \right\}, \tag{3.7}$$

where (x_i, y_i) are the source points of RBF.

Based on (A1),(A2), we may choose the admissible functions,

$$v = \sum_{i=1}^{L} \tilde{a}_i g_i(x,y), \qquad \text{in } S, \tag{3.8}$$

where $\tilde{a}_i$ are unknown coefficients to be sought. Denote by V_h, the finite dimensional collection of the admissible functions (3.8) To solve (3.1)–(3.3), the Ritz-Galerkin method can be written as follows. To seek solution u_L, such that

$$b(u_L, v) = f(v), \qquad \forall v \in V_h, \tag{3.9}$$

where

$$b(u,v) = \iint_S \Delta u \Delta v + \int_{\Gamma_N} u_\nu v_\nu + \int_{\Gamma_R} (u_\nu + \beta u)(v_\nu + \beta v), \tag{3.10}$$

$$f(v) = -\iint_S f\Delta v + \int_{\Gamma_N} q_1 v_\nu + \int_{\Gamma_R} q_2 (v_\nu + \beta v). \tag{3.11}$$

When we view the collocation method as a Ritz-Galerkin method involving approximation quadratures, the radial basis collocation method (RBCM) can be written as follows. To seek solution $\hat{u}_L$, such that

$$\hat{b}(\hat{u}_L, v) = \hat{f}(v), \qquad \forall v \in V_h, \tag{3.12}$$

where

$$\hat{b}(u,v) = \widehat{\iint}_S \Delta u \Delta v + \widehat{\int}_{\Gamma_N} u_\nu v_\nu + \widehat{\int}_{\Gamma_R} (u_\nu + \beta u)(v_\nu + \beta v), \tag{3.13}$$

$$\hat{f}(v) = -\widehat{\iint}_S f\Delta v + \widehat{\int}_{\Gamma_N} q_1 v_\nu + \widehat{\int}_{\Gamma_R} q_2 (v_\nu + \beta v), \tag{3.14}$$

where $\widehat{\iint}_S$, $\widehat{\int}_{\Gamma_N}$, and $\widehat{\int}_{\Gamma_R}$ denote the approximations of $\iint_S$, $\int_{\Gamma_N}$, and $\int_{\Gamma_R}$ by some integration rules, respectively. We may choose the Newton-Cotes rules or the Legendre-Gauss rules,

$$\widehat{\iint}_S F = \sum_{ij} \alpha_{ij} F(Q_{ij}), \qquad Q_{ij} \in S, \tag{3.15}$$

$$\widehat{\int}_\Gamma F = \sum_i \alpha_i F(Q_i), \qquad Q_i \in \Gamma, \tag{3.16}$$

where α_{ij} and α_i are positive weights. Then, we can formulate the collocation equations directly at Q_{ij} and Q_i as follows.

$$\sqrt{\alpha_{ij}}\,(\Delta v + f)(Q_{ij}) = 0, \qquad Q_{ij} \in S, \tag{3.17}$$

$$\sqrt{\alpha_i^N}\,(v_\nu - q_1)(Q_i) = 0, \qquad Q_i \in \Gamma_N, \tag{3.18}$$

$$\sqrt{\alpha_i^R}\,(v_\nu + \beta v - q_2)(Q_i) = 0, \qquad Q_i \in \Gamma_R. \tag{3.19}$$

Equations (3.17)–(3.19) in $S \cup \Gamma_N \cup \Gamma_R$ are written simply as

$$F\vec{x} = \vec{b}, \tag{3.20}$$

where $\vec{x}$ is a vector consisting of $\tilde{a}_i$, $\vec{b}$ is a known vector, matrix $F \in R^{N_c \times L}$, and N_c is the number of collocation nodes in $S \cup \Gamma_N \cup \Gamma_R$. In this paper, let $N_c \geq L$, and we may seek the solutions of the entire RBCM by the least squares method (LSM) in Golub and Loan [19].

Now, we would like to provide the error estimates for solution $\hat{u}_L$ in (3.12) by following some schemes of the FEM theory, the reader may refer to [18] for more details. Denote the space

$$H^* = \left\{v, v \in L^2(S),\ v \in H^1(S),\ \Delta v \in L^2(S)\right\}, \tag{3.21}$$

accompanied by the norm

$$\|v\|_h = \left\{\|v\|_{1,S}^2 + \|\Delta v\|_{0,S}^2 + \|v_\nu\|_{0,\Gamma_N}^2 + \|(v_\nu + \beta v)\|_{0,\Gamma_R}^2\right\}^{1/2}, \tag{3.22}$$

where $\|v\|_{1,S}$ is the Sobolev norm. In order to derive our main theorem, Theorem 3.1 given later, the following lemmas are needed.

LEMMA 3.1. *There exist two inequalities*

$$b(u,v) \leq C\|u\|_h \times \|v\|_h, \qquad \forall v \in V_h, \tag{3.23}$$

$$b(v,v) \geq C_0\|v\|_h^2, \qquad \forall v \in V_h, \tag{3.24}$$

where C_0 and C are two positive constants independent on L.

PROOF. From equation (3.10), we have

$$\begin{aligned} b(u,v) &\leq C_1\sqrt{\iint_S (\Delta u)^2}\sqrt{\iint_S (\Delta v)^2} + C_2\sqrt{\int_{\Gamma_N}(u_\nu)^2}\sqrt{\int_{\Gamma_N}(v_\nu)^2} \\ &\quad + C_3\sqrt{\int_{\Gamma_R}(u_\nu+\beta u)^2}\sqrt{\int_{\Gamma_R}(v_\nu+\beta v)^2} \\ &\leq C\Big\{\|\Delta u\|_{0,S}\|\Delta v\|_{0,S} + \|u_\nu\|_{0,\Gamma_N}\|v_\nu\|_{0,\Gamma_N} \\ &\quad + \|u_\nu+\beta u\|_{0,\Gamma_R}\|v_\nu+\beta v\|_{0,\Gamma_R} + \|u\|_{1,S}\|v\|_{1,S}\Big\} \\ &\leq C\left\{\|\Delta u\|_{0,S}^2 + \|u_\nu\|_{0,\Gamma_N}^2 + \|u_\nu+\beta u\|_{0,\Gamma_R}^2 + \|u\|_{1,S}^2\right\}^{1/2} \\ &\quad \times \left\{\|\Delta v\|_{0,S}^2 + \|v_\nu\|_{0,\Gamma_N}^2 + \|v_\nu+\beta v\|_{0,\Gamma_R}^2 + \|v\|_{1,S}^2\right\}^{1/2} \\ &\leq C\|u\|_h \times \|v\|_h, \end{aligned}$$

where C_i and C are generic constants, and their values may be different in different contexts. The first desired result (3.23) is obtained.

Next, from Green's formula, we have

$$\begin{aligned} |v|_{1,S}^2 &= \iint_S \nabla v^2 = -\iint_S v\Delta v + \int_{\partial S} v_\nu v \\ &\leq \|\Delta v\|_{0,S}\|v\|_{0,S} + \int_{\Gamma_N} v_\nu v + \int_{\Gamma_R}(v_\nu+\beta v)\,v - \int_{\Gamma_R}\beta v^2 \\ &\leq \left\{\|\Delta v\|_{0,S} + \|v_\nu\|_{0,\Gamma_N} + \|v_\nu+\beta v\|_{0,\Gamma_R}\right\}\|v\|_{1,S} - \int_{\Gamma_R}\beta v^2, \end{aligned} \tag{3.25}$$

where $\partial S = \Gamma = \Gamma_N \cup \Gamma_R$, and two bounds are used,

$$\|v\|_{0,\Gamma_N} \le C \|v\|_{1,S}, \qquad \|v\|_{0,\Gamma_R} \le C \|v\|_{1,S}.$$

Besides, from (3.25), we obtain

$$\|v\|_{1,S}^2 \le C \left(|v|_{1,S}^2 + \int_{\Gamma_R} \beta v^2 \right) \le C \left\{ \|\Delta v\|_{0,S} + \|v_\nu\|_{0,\Gamma_N} + \|v_\nu + \beta v\|_{0,\Gamma_R} \right\} \|v\|_{1,S}.$$

This leads to

$$\|v\|_{1,S} \le C \left\{ \|\Delta v\|_{0,S} + \|v_\nu\|_{0,\Gamma_N} + \|v_\nu + \beta v\|_{0,\Gamma_R} \right\},$$

and then,

$$\|v\|_{1,S}^2 \le C \left\{ \|\Delta v\|_{0,S}^2 + \|v_\nu\|_{0,\Gamma_N}^2 + \|v_\nu + \beta v\|_{0,\Gamma_R}^2 \right\} = Cb(v,v). \tag{3.26}$$

Moreover, from (3.26) and (3.10), we obtain

$$\begin{aligned} b(v,v) &= \frac{1}{2} b(v,v) + \frac{1}{2} b(v,v) \\ &\ge C_0 \|v\|_{1,S}^2 + \frac{1}{2} \left\{ \|\Delta v\|_{0,S}^2 + \|v_\nu\|_{0,\Gamma_N}^2 + \|v_\nu + \beta v\|_{0,\Gamma_R}^2 \right\} \\ &\ge \bar{C}_0 \|v\|_h^2, \end{aligned}$$

where $\bar{C}_0 = \min\{1/2, C_0\}$. The second desired result (3.24) is obtained. This completes the proof of Lemma 3.1. ∎

We make one more assumption.

(A3) Suppose that, for v in (3.8) there exists positive constant C, such that

$$\|v\|_{k,S} \le CL^k \|v\|_{0,S}, \qquad v \in V_h, \tag{3.27}$$

$$\|v\|_{k,\Gamma} \le CL^k \|v\|_{0,\Gamma}, \qquad v \in V_h, \tag{3.28}$$

$$\|v_\nu\|_{k,\Gamma} \le CL^{k+1} \|v\|_{0,\Gamma}, \qquad v \in V_h, \tag{3.29}$$

where ν is unit outward normal to Γ.

The inverse inequalities in (3.27)–(3.29) are crucial to following lemmas and theorem. The proof is new to [2,3], and it is deferred to Section 4.

LEMMA 3.2. *For the rules (3.15),(3.16) with order r, there exist the bounds, for $v \in V_h$,*

$$\left| \left(\iint_S - \widehat{\iint}_S \right) (\Delta v)^2 \right| \le CH^{(r+1)} L^{(r+3)} \|v\|_{1,S}^2, \tag{3.30}$$

$$\left| \left(\int_{\Gamma_N} - \widehat{\int}_{\Gamma_N} \right) (v_\nu)^2 \right| \le CH^{(r+1)} L^{(r+3)} \|v\|_{1,S}^2, \tag{3.31}$$

$$\left| \left(\int_{\Gamma_R} - \widehat{\int}_{\Gamma_R} \right) (v_\nu + \beta v)^2 \right| \le CH^{(r+1)} L^{(r+3)} \|v\|_{1,S}^2, \tag{3.32}$$

where H denotes the maximal spacing between integration nodes, i.e., collocation points, and L denotes the number of RBF

PROOF. Choose the integration rules of order r,

$$\widehat{\int}_\Gamma g = \int_\Gamma \hat{g},$$

where $\hat{g}$ is the interpolant polynomial of order r on the partition of Γ with the maximal meshspacing H. Then, we obtain

$$\left|\left(\int_{\Gamma}-\widehat{\int}_{\Gamma}\right)g\right|=\left|\int_{\Gamma}g-\int_{\Gamma}\hat{g}\right|=\left|\int_{\Gamma}(g-\hat{g})\right|\leq CH^{r+1}\left|g\right|_{r+1,\Gamma}.$$

Based on (A3), letting $g=(v_\nu)^2$ and $\Gamma=\Gamma_N$, we have

$$\begin{aligned}|g|_{r+1,\Gamma}=\left|(v_\nu)^2\right|_{r+1,\Gamma_N}&\leq C\sum_{i=0}^{r+1}|v_\nu|_{r+1-i,\Gamma_N}|v_\nu|_{i,\Gamma_N}\\&\leq C\sum_{i=0}^{r+1}\left(L^{r+2-i}\|v\|_{1,S}\times L^{i+1}\|v\|_{1,S}\right)\leq CL^{r+3}\|v\|_{1,S}^2.\end{aligned}$$

Combining above two inequalities gives the desired result (3.31). Equations (3.30) and (3.32) can be similarly derived. ∎

LEMMA 3.3. *Let Lemmas 3.1 and 3.2 hold. We choose H to satisfy*

$$H^{r+1}L^{r+3}=o(1), \tag{3.33}$$

then, there exists the uniform V_h-elliptic inequality

$$\hat{b}(v,v)\geq C\|v\|_h^2,\qquad\forall\, v\in V_h. \tag{3.34}$$

PROOF. From (3.24) and Lemma 3.2, we have

$$\begin{aligned}\hat{b}(v,v)&\geq b(v,v)-CH^{r+1}L^{r+3}\|v\|_{1,S}^2\\&\geq C_0\|v\|_h^2-CH^{r+1}L^{r+3}\|v\|_{1,S}^2\\&\geq C_0\left\{\left(1-\frac{C}{C_0}H^{r+1}L^{r+3}\right)\|v\|_{1,S}^2+\|\Delta v\|_{0,S}^2+\|v_\nu\|_{0,\Gamma_N}^2+\|v_\nu+\beta v\|_{0,\Gamma_R}^2\right\}\\&\geq\frac{C_0}{2}\|v\|_h^2.\end{aligned}$$

Let $C=C_0/2$, and this completes the proof of Lemma 3.3. ∎

We can obtain an important theorem as follows.

THEOREM 3.1. *Suppose that there exist two inequalities*

$$\hat{b}(u,v)\leq C\|u\|_h\times\|v\|_h,\qquad\forall\, v\in V_h, \tag{3.35}$$

$$\hat{b}(v,v)\geq C_0\|v\|_h^2,\qquad\forall\, v\in V_h, \tag{3.36}$$

where C_0 and C are two positive constants independent on L. Then, when choosing H and L as (3.33), the solution of the RBCM in (3.12) has the error bound,

$$\|u-\hat{u}_L\|_h=C\inf_{v\in V_h}\|u-v\|_h\leq C\left\{\|R_L\|_{2,S}+\|(R_L)_\nu\|_{0,\Gamma_N}+\|(R_L)_\nu\|_{0,\Gamma_R}\right\}, \tag{3.37}$$

where C is a constants independent on L.

PROOF. Since u is true solution, we have $b(u,v)=f(v)$. By using Lemma 3.2, we can derive

$$\hat{b}(u,v)\leq\hat{f}(v)+CH^{r+1}L^{r+3}\|v\|_{1,S}^2,\qquad\forall\, v\in V_h.$$

Since $\hat{u}_L$ is the solution of RBCM, we have

$$\hat{b}\left(\hat{u}_L, v\right) = \hat{f}\left(v\right), \qquad \forall v \in V_h,$$

and then,

$$\hat{b}\left(u - \hat{u}_L, v\right) \le CH^{r+1}L^{r+3}\left\|v\right\|_{1,S}^2, \qquad \forall v \in V_h.$$

Let $w = \hat{u}_L - v \in V_h$, from the above bound, we obtain

$$\begin{aligned} C_0\left\|w\right\|_h^2 &\le \hat{b}\left(\hat{u}_L - v, w\right) = \hat{b}\left(\hat{u}_L - u + u - v, w\right) \\ &\le \left\|u - v\right\|_h \left\|w\right\|_h + CH^{r+1}L^{r+3}\left\|v\right\|_{1,S}^2 \\ &\le \left\|u - v\right\|_h \left\|w\right\|_h + CH^{r+1}L^{r+3}\left\|w\right\|_{1,S}^2. \end{aligned}$$

This leads to

$$\left\{C_0 - CH^{r+1}L^{r+3}\right\}\left\|w\right\|_h^2 \le \left\|u - v\right\|_h \left\|w\right\|_h.$$

Moreover, from (3.33), we have

$$\left\|\hat{u}_L - v\right\|_h = \left\|w\right\|_h \le \frac{1}{\left(C_0 - C \times o(1)\right)}\left\|u - v\right\|_h \le C_1\left\|u - v\right\|_h$$

From triangle inequality, we have

$$\left\|u - \hat{u}_L\right\|_h \le \left\|u - v\right\|_h + \left\|\hat{u}_L - v\right\|_h \le C\left\|u - v\right\|_h,$$

and then,

$$\left\|u - \hat{u}_L\right\|_h = C \inf_{\forall v \in V_h}\left\|u - v\right\|_h \le C\left\{\left\|R_L\right\|_{1,S} + \left\|\Delta R_L\right\|_{0,S} + \left\|(R_L)_\nu\right\|_{0,\Gamma_N} + \left\|(R_L)_\nu\right\|_{0,\Gamma_R}\right\}.$$

This completes the proof of Theorem 3.1. ∎

REMARK 3.1. Theorem 3.1 implies that the errors of the solutions for Poisson's equation using RBCM are bounded by the truncation errors of RBF multiplied by a constant. From Assumption (A2), we assure that the errors of solution $\hat{u}_L$ have exponential convergence. Moreover, equation (3.33) has the following relation,

$$HL^{(1+2/(r+1))} = o\left(1\right), \tag{3.38}$$

where H and L are given in Lemma 3.2. Then, we take $L = N^2$, and assume that the radial distance δ defined in (3.7) satisfies the relation $\delta \approx O(1/N)$. Furthermore, we obtain an important relation,

$$H = o\left(\delta^{(2+4/(r+1))}\right). \tag{3.39}$$

From (3.39), we know how to balance the collocation nodes and the source points of RBF.

REMARK 3.2. In this paper, the Newton-Cotes rules of integration are chosen for simplicity in exposition. From equation (3.39), we need a dense set of integration nodes in order to satisfy the V_h-elliptic inequality, and then, to obtain the exponential convergence rates. The solution domain may not be confined in rectangles (or boxes for three dimensions). When the collocation methods using the radial basis functions are applied to solving PDEs in three dimensions, the simplest central rule may also be used. An integral in a closed region of three dimensions is approximated by the value of the integrand at the center of gravity (or roughly at any point) of the region times the volume of the region Hence, the integration quadrature is not a severe problem in the collocation methods described in this paper.

3.2. Combination of FEM and RBCM

Consider Poisson's equation with the Dirichlet condition,

$$-\Delta u = f(x,y), \qquad \text{in } S, \tag{3.40}$$
$$u = 0, \qquad \text{on } \Gamma, \tag{3.41}$$

where S is a polygon, and Γ is its boundary. Let S be divided by Γ_0 into two disjoint subregions, S_1 and S_2 (i.e., $S = S_1 \cup S_2 \cup \Gamma_0$ and $S_1 \cap S_2 = \emptyset$). On the interior boundary Γ_0, there hold the interior continuity conditions,

$$u^+ = u^-, \quad u_\nu^+ = u_\nu^-, \qquad \text{on } \Gamma_0, \tag{3.42}$$

where $u_\nu = \frac{\partial u}{\partial \nu}$, $u^+ = u$ on $\Gamma_0 \cup S_2$, and $u^- = u$ on $\Gamma_0 \cup S_1$. Assume that the solution u in S_2 is smoother than u in S_1. We choose the finite-element method (FEM) in S_1 and the Ritz-Galerkin method in S_2, whose discrete forms lead to the CM. Let S_1 be partitioned into small triangles: $\triangle_{ij}$, i.e., $S_1 = \bigcup_{ij} \triangle_{ij}$. Denote by h_{ij} the boundary length of $\triangle_{ij}$. The $\triangle_{ij}$ are said to be quasiuniform if $h/\min\{h_{ij}\} \le C$, where $h = \max_{\triangle_{ij} \subset S_1}\{h_{ij}\}$, and C is a constant independent of h. Then, the admissible functions may be expressed by

$$v = \begin{cases} v^- = v_k, & \text{in } S_1, \\ v^+ = \sum\limits_{i=1}^{L} \tilde{a}_i g_i(x,y), & \text{in } S_2, \end{cases} \tag{3.43}$$

where $\tilde{a}_i$ are unknown coefficients, and v_k are piecewise Lagrange polynomials of power k in S_1 in the FEM. Assume that (A1),(A2) hold in S_2, and $g_i(x,y) \in C^2(S_2 \cup \partial S_2)$ are the RBF, so that $v^+ \in C^2(S_2 \cup \partial S_2)$. Therefore, we may evaluate (3.40) directly from

$$\left(\Delta v^+ + f\right)(Q_{ij}) = 0, \qquad \text{for } Q_{ij} \in S_2, \tag{3.44}$$

at certain collocation nodes $Q_{ij} \in S_2$. Note that v in (3.43) is not continuous on the interior boundary Γ_0. Hence, to satisfy (3.42), the interior collocation equations are obtained as follows,

$$v^+(Q_i) = v^-(Q_i), \qquad \text{for } Q_i \in \Gamma_0, \tag{3.45}$$
$$v_\nu^+(Q_i) = v_\nu^-(Q_i), \qquad \text{for } Q_i \in \Gamma_0. \tag{3.46}$$

Note that equations (3.44)–(3.46) are straightforward and easy to be formulated.

Denote by V_h^0 the finite-dimensional collection of (3.43) satisfying $v|_\Gamma = 0$, where we simply assume $g_i(x,y)|_{\partial S_2 \cap \Gamma} = 0$. If such a condition does not hold, the corresponding collocation equations on $\partial S_2 \cap \Gamma$ are also needed, and the arguments can be provided similarly.

The combination of the FEM-RBCM involving integration approximation is given by as follows. To seek the approximation solution $\hat{u}_h \in V_h^0$, such that

$$\hat{a}^*(\hat{u}_h, v) = f_1(v), \qquad \forall v \in V_h^0, \tag{3.47}$$

where

$$\begin{aligned} \hat{a}^*(u,v) = {}& \iint_{S_1} \nabla u \nabla v + \int_{\Gamma_0} u_\nu^- v^- + P_c \widehat{\iint_{S_2}} (\Delta u + f)(\Delta v + f) \\ & + \frac{P_c}{h} \widehat{\int_{\Gamma_0}} \left(u^+ - u^-\right)\left(v^+ - v^-\right) + P_c \widehat{\int_{\Gamma_0}} \left(u_\nu^+ - u_\nu^-\right)\left(v_\nu^+ - v_\nu^-\right), \end{aligned} \tag{3.48}$$

and

$$f_1(v) = \iint_{S_1} f v, \tag{3.49}$$

where $\nabla u = u_x\vec{i} + u_y\vec{j}$, $u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$, $u_\nu = \frac{\partial u}{\partial \nu}$, and ν is the unit outward normal to ∂S_2. h is the maximal boundary length of Δ_{ij} or $\square_{ij}$ in S_1, and $P_c > 0$ is chosen to be suitably large but still independent on h.

Now, let us establish the linear algebraic equations of combinations (3.47) of FEM-RBCM. First, considering the FEM in S_1 only, we have

$$a_1(\hat{u}_h, v) = f_1(v), \qquad \forall v \in V_h, \tag{3.50}$$

where

$$a_1(u,v) = \iint_{S_1} \nabla u \nabla v + \int_{\Gamma_0} u_\nu^- v^-, \qquad f_1(v) = \iint_{S_1} fv. \tag{3.51}$$

By means of a traditional procedure of finite element method [18], we obtain the linear algebraic equations,

$$A_1\vec{x}_1 = \vec{b}_1, \tag{3.52}$$

where $\vec{x}_1$ is merely a vector consisting of v_{ij}, and matrix A_1 is nonsymmetric.

Next, we choose the integration rules (Newton-Cotes rules or Legendre-Gauss rules) in S_2. We may formulate collocation equations at $Q_{ij} \in S_2$, and $Q_i \in \Gamma_0$ directly. The collocation equations at Q_{ij}, and Q_i are given by

$$\sqrt{P_c\alpha_{ij}}\left(\Delta v^+ + f\right)(Q_{ij}) = 0, \qquad Q_{ij} \in S_2, \tag{3.53}$$

$$\sqrt{\frac{P_c\alpha_i}{2h}}\left(v^+ - v^-\right)(Q_i) = 0, \qquad Q_i \in \Gamma_0, \tag{3.54}$$

$$\sqrt{\frac{P_c\alpha_i}{2h}}\left(v_\nu^+ - v_\nu^-\right)(Q_i) = 0, \qquad Q_i \in \Gamma_0, \tag{3.55}$$

where $v_\nu^-(Q_i) = (v_{1i} - v_{0i})/h$, $v_{0i} = v(Q_i)$, and v_{1i} are the nodal variables in S_1 normal to Γ_0. Equations (3.53)–(3.55) in $S_2 \cup \Gamma_0$ are denoted by

$$A_2\vec{x}_2 = \vec{b}_2, \tag{3.56}$$

where $\vec{x}_2$ is a vector consisting of $\tilde{a}_i$, v_{1i}, and v_{0i}. v_{0i} and v_{1i} are the unknowns on the two boundary layer nodes in S_1 close to Γ_0. Denote by M_C the number of all collocation nodes in S_2 and ∂S_2, and by N_B the number of v_{1i} and v_{0i}. The matrix $A_2 \in R^{M_C\times(L+N_B)}$. Therefore, we can see

$$\begin{aligned}\frac{P_c}{2}\widehat{\iint_{S_2}}(\Delta v + f)^2 + \frac{P_c}{2h}\widehat{\int_{\Gamma_0}}\left(v^+ - v^-\right)^2 + \frac{P_c}{2}\widehat{\int_{\Gamma_0}}\left(v_\nu^+ - v_\nu^-\right)^2 \\ = \frac{1}{2}\vec{x}_2^\top A_2^\top A_2\vec{x}_2 - A_2^\top\vec{b}_2\vec{x}_2 + \vec{c}.\end{aligned} \tag{3.57}$$

Combining (3.52) and (3.57) yields explicitly the following,

$$A\vec{x} = \vec{b}, \tag{3.58}$$

$$A = A_1 + A_2^\top A_2, \qquad \vec{b} = \vec{b}_1 + A_2^\top\vec{b}_2, \tag{3.59}$$

where $\vec{x}$ is a vector consisting of the coefficients $\tilde{a}_i$ and v_{ij} in $S_1 \cup \Gamma_0$. Denote by N_E, the number of nodes on $S_1 \cup \Gamma_0$. Then, vector $\vec{x}$ in (3.58) has $N_E + L$ dimensions. When P_c is chosen large enough, matrix $A \in R^{(L+N_E)\times(L+N_E)}$ in (3.58) is positive definite, nonsymmetric, and sparse when $N_E \gg L$. When $L + N_E$ is not huge, we may choose the Gaussian elimination without pivoting to obtain $\vec{x}$ from (3.58), see [19].

Now, we give error bounds for the solution $\hat{u}_h$ from (3.47) whose proofs are reported in [2]. The combination (3.47) can be described equivalently as follows. To seek $\hat{u}_h \in V_h^0$, such that

$$\hat{a}(\hat{u}_h, v) = \hat{f}(v), \qquad \forall v \in V_h^0, \tag{3.60}$$

where

$$\hat{a}(u,v) = \iint_{S_1} \nabla u \nabla v + \int_{\Gamma_0} u_n^- v^- + P_c \widehat{\iint_{S_2}} \Delta u \, \Delta v + \frac{P_c}{h} \widehat{\int_{\Gamma_0}} \left(u^+ - u^-\right)\left(v^+ - v^-\right) + P_c \widehat{\int_{\Gamma_0}} \left(u_\nu^+ - u_\nu^-\right)\left(v_\nu^+ - v_\nu^-\right), \tag{3.61}$$

$$\hat{f}(v) = \iint_{S_1} fv - P_c \widehat{\iint_{S_2}} f \Delta v. \tag{3.62}$$

Denote the space,

$$H^{**} = \left\{v,\ v \in L^2(S),\ v \in H^1(S_1),\ v \in H^1(S_2),\ \Delta v \in L^2(S_2),\ \text{and } v\,|_{\Gamma} = 0\right\}, \tag{3.63}$$

accompanied with the norm,

$$|||v||| = \left(\|v\|_{1,S_1}^2 + P_c \|v\|_{1,S_2}^2 + P_c \|\Delta v\|_{0,S_2}^2 + \frac{P_c}{h} \left\|v^+ - v^-\right\|_{0,\Gamma_0}^2 + P_c \left\|v_\nu^+ - v_\nu^-\right\|_{0,\Gamma_0}^2 \right)^{1/2}, \tag{3.64}$$

where

$$\|v\|_1 = \left\{ \|v\|_{1,S_1}^2 + \|v\|_{1,S_2}^2 \right\}^{1/2}, \qquad |v|_1 = \left\{ |v|_{1,S_1}^2 + |v|_{1,S_2}^2 \right\}^{1/2}, \tag{3.65}$$

and $\|v\|_{1,S_1}$ and $\|v\|_{1,S_2}$ are Sobolev norms. Obviously, $V_h^0 \subset H^{**}$. We obtain an important theorem below, and an outline of the proofs is given in Remark 3.3.

THEOREM 3.2. *Suppose that there hold the inequalities,*

$$\hat{a}(u,v) \le C|||u||| \times |||v|||, \qquad \forall v \in V_h^0, \tag{3.66}$$

$$\hat{a}(v,v) \ge C_0 |||v|||^2, \qquad \forall v \in V_h^0, \tag{3.67}$$

where $C_0 > 0$ and C are two constants independent on h and L. Then, the solution of combination (3.47) has the error bound,

$$|||u - \hat{u}_h||| \le C \inf_{v \in V_h} |||u - v|||. \tag{3.68}$$

We can obtain the following corollary from Theorem 3.2 (see [2]).

COROLLARY 3.1. *Let all conditions in Theorem 3.2 hold. Suppose that*

$$u \in H^{k+1}(S_1) \qquad \text{and} \qquad u \in H^{k+1}(\Gamma_0). \tag{3.69}$$

Then, there exists the error bound,

$$|||u - \hat{u}_h||| \le C \left\{ h^k |u|_{k+1,S_1} + \sqrt{P_c} \|R_L\|_{2,S_2} + \sqrt{P_c} \left(h^{k+1/2} |u|_{k+1,\Gamma_0} + \frac{1}{\sqrt{h}} \|R_L\|_{0,\Gamma_0} + \|(R_L)_\nu\|_{0,\Gamma_0} \right) \right\}. \tag{3.70}$$

Also, suppose that the number L in (3.43) is chosen, such that

$$\|R_L\|_{2,S_2} = O\left(h^k\right), \qquad \|R_L\|_{0,\Gamma_0} = O\left(h^{k+1/2}\right), \qquad \|(R_L)_\nu\|_{0,\Gamma_0} = O\left(h^k\right). \tag{3.71}$$

Then, the optimal convergence rate is given by,

$$|||u - \hat{u}_h||| = O\left(h^k\right). \tag{3.72}$$

REMARK 3.3 Let us give an outline of the proofs in [2] for the V_h^0-elliptic inequality (3.67), which consists of two steps. Step I for the original equation of (3.61) without integration approximation, we need to prove $a(v,v) \geq C_0 |||v|||^2$, $\forall v \in V_h^0$ Step II for the Newton-Cotes integration rules of order r, assume that the inverse inequalities in (A3) hold, then, we obtain

$$hL^2 = o(1), \qquad HL^{(1+2/(r+1))} = o(1), \tag{3.73}$$

where $h = \max_{ij}\{h_{ij}\}$ in S_1, and H denotes the maximal spacing between integration nodes, i.e., collocation nodes, and L denotes the number of RBF in S_2. When we take $L = N^2$, and assume that the radial distance δ defined in (3.7) satisfies relation $\delta \approx O(1/N)$, then, we have the relations

$$h = o\left(\delta^4\right), \qquad H = o\left(\delta^{(2+4/(r+1))}\right). \tag{3.74}$$

From (3.74), we also know how to balance the mesh of FEM and the source points of RBF.

REMARK 3.4. From those error analyses, we discover that the integration quadrature plays a role only for satisfying the uniformly V_h^0-elliptic inequalities (3.36) and (3.67), but not for improving the accuracy of the solutions. As long as the maximal spacing H between collocation nodes is small enough, there always exist the optimal orders of solution errors from the radial basis collocation methods and their combinations.

4. INVERSE ESTIMATES FOR RADIAL BASIS FUNCTIONS

Although, there exist many papers concerning RBF, only a few of them are related to solutions of partial differential equations (see [8,9,16]). In the Ritz-Galerkin method (RGM) [1], orthogonal polynomials and particular solutions are chosen, that have been replaced by RBF in Section 3.

Recently, theoretical framework for the collocation methods has been established by that Hu and Li [2,3] which can be easily applied to RBCMs and their combinations, except the crucial inverse estimates (3.27)–(3.29) of RBF need to be proven. Note that our results in [2,3] and, in this paper, are more comprehensive than those in [9] and [16], because various boundary conditions are also involved, and because the combined methods are developed. Below, let us explore the inverse estimates (3.27)–(3.29) needed for the RBF. Take the multiquadric functions as example,

$$Q_L(x,y) = a_0 + \sum_{i=1}^{L} a_i g_i(x,y), \tag{4.1}$$

where $g_i(x,y) = (r_i^2 + c^2)^{1/2}$. To approximate function $f(x,y)$, the collocation equations are given by

$$Q_L(x_i,y_i) = f_i(x,y), \qquad i = 1,2,\ldots,L, \quad \sum_{i=1}^{L} a_i = 0. \tag{4.2}$$

Equation (3.27) is essential, because equations (3.28) and (3.29) can be easily derived from (or replaced by) (3.27) (see [3]). Hence, we focus on the proof of (3.27) for the multiquadratic functions. We have the following lemma.

LEMMA 4.1. *Let $f(x,y) \in S$, S be a rectangle, and $c \geq 1$. There exists $\lambda \in (0,1)$ independent on f, c, and δ such that, for sufficiently small δ, there exist the bounds,*

$$\|f - Q_L\|_{0,S} \leq C \exp\left(\alpha c^2\right) \lambda^{c/\delta} \|f\|_{0,S}, \tag{4.3}$$

$$\|f - Q_L\|_{k,S} \leq C\delta^c \|f\|_{0,S}, \tag{4.4}$$

where $\alpha = \epsilon_h^2\sigma/2$, ϵ_h, and σ are two real constants in the Fourier transform, and C is a constant independent on δ and α.

PROOF. Based on Madych [5, Theorem 1, cf. (10), p. 124], there exists a bound,

$$\|f - Q_L\|_{0,\infty,S} \leq C \exp\left(\alpha c^2\right) \lambda^{c/\delta} \|f\|_{0,S}, \qquad \alpha = \frac{\epsilon_h^2 \sigma}{2}. \tag{4.5}$$

Since S is a bounded domain, we have

$$\|f - Q_L\|_{0,S} \le C \|f - Q_L\|_{0,\infty,S} \le C \exp\left(\alpha c^2\right) \lambda^{c/\delta} \|f\|_{0,S} . \tag{4.6}$$

This is the first result (4.3).

Next, we also obtain from [5, Theorem 4, p. 127],

$$\|f - Q_L\|_{k,S} \le C\delta^c \|f\|_{C_{h_c}} , \tag{4.7}$$

where the norm is defined by [5, cf. (6) p. 124],

$$\|f\|^2_{C_{h_c}} = \sum_{i=1}^{2} \iint_S \left|\xi_i \hat{f}(\xi)\right|^2 \left(|\xi|^2 h_c(\xi)\right)^{-1} d\xi, \tag{4.8}$$

where $h_c = \sqrt{r^2 + c^2}$, and the Fourier transform,

$$\hat{f}(x) = \iint_S f(x) \exp\left(-i(x,\xi)\right) dx. \tag{4.9}$$

Since $h_c \ge h_1$, for $c \ge 1$, we have

$$\|f\|_{C_{h_c}} \le \|f\|_{C_{h_1}} , \qquad c \ge 1. \tag{4.10}$$

Moreover, there is an estimate in [5, p. 124] that, for $g(x) = f(cx)$,

$$\|g\|^2_{C_{h_1}} \le C \exp\left(2\epsilon_h \sigma c\right) \|f\|^2_{0,S} , \tag{4.11}$$

where ϵ_h and σ are constants. Hence, if $c = 1$ and $g(x) = f(x)$, we have from (4.11)

$$\|f\|^2_{C_{h_1}} \le C \exp\left(2\epsilon_h \sigma\right) \|f\|^2_{0,S} . \tag{4.12}$$

Combining (4.7), (4.10), and (4.12) yields

$$\|f - Q_L\|_{k,S} \le C\delta^c \|f\|_{C_{h_1}} \le C \exp\left(\epsilon_h \sigma\right) \delta^c \|f\|_{0,S} \le C_1 \delta^c \|f\|_{0,S} , \tag{4.13}$$

where $C_1 = C\exp(\epsilon_h\sigma)$. This is the second result (4.4), and completes the proof of Lemma 4.1. ∎

Lemma 4.1 and [5] imply that the approximate solutions have the exponential convergence rate $O(\lambda^{c/\delta})$ with respect to δ, but the derivatives of low order have only polynomial convergence rates $O(\delta^c)$. Although (4.4) is very conservative, being the linear convergence rate $O(\delta)$, for $c \ge 1$, it is used as reasonable assumptions (4.15) leading to the inverse estimates shown below. Suppose that there exist the bounds,

$$\|f - Q_L\|_{0,S} \le o(1) \|f\|_{0,S} , \tag{4.14}$$

$$\|f - Q_L\|_{k,S} \le C \|f\|_{0,S} , \qquad k = 1, 2, \dots , \tag{4.15}$$

where $o(1) \ll 1$, and C is a bounded constant as $\delta \to 0$ Denote polynomials f by

$$f = f_N(x,y) = \sum_{i,j=0}^{N} b_{ij} x^i y^j . \tag{4.16}$$

We cite a Lemma from [1,3].

LEMMA 4.2. *For the polynomials of order N defined in S, there exists a constant independent of N, such that*

$$\|f_N\|_{k,S} \le CN^{2k} \|f_N\|_{0,S} . \tag{4.17}$$

Let (4.2) be given. For uniquely determining the polynomials (4.16), we choose

$$N^2 < L \le (N+1)^2 , \tag{4.18}$$

and employ more collocation equations

$$Q_L(x_i, y_i) = f_i, \qquad i = 1, 2, \dots, (N+1)^2, \quad \sum_{i=1}^{(N+1)^2} a_i = 0, \tag{4.19}$$

where the first L collocation equations are the same as in (4.2). Hence, the errors of the solutions and their derivatives would not decrease, and Lemma 4.1 also holds. Then, we may still assume (4.14) and (4.15), and prove the following theorem.

THEOREM 4.1. *Let (4.14) and (4.15) hold for f_N satisfying (4.16) and (4.19), where S is a rectangle. Then, there exists the bound,*

$$\|Q_L\|_{k,S} \le CL^k \|Q_L\|_{0,S} . \tag{4.20}$$

PROOF. From the triangle inequality, we have

$$\|Q_L\|_{k,S} \le \|f_N\|_{k,S} + \|Q_L - f_N\|_{k,S} . \tag{4.21}$$

From Lemma 4.2 and (4.15),

$$\|Q_L\|_{k,S} \le CN^{2k} \|f_N\|_{0,S} + C\|f_N\|_{0,S} \le CN^{2k} \|f_N\|_{0,S} . \tag{4.22}$$

Similarly, from (4.14) we have

$$\|f_N\|_{0,S} \le \|Q_L\|_{0,S} + \|Q_L - f_N\|_{0,S} \le \|Q_L\|_{0,S} + o(1) \|f_N\|_{0,S} . \tag{4.23}$$

This leads to

$$\|f_N\|_{0,S} \le \frac{1}{(1 - o(1))} \|Q_L\|_{0,S} \le \frac{1}{2} \|Q_L\|_{0,S} . \tag{4.24}$$

Since $N \le \sqrt{L}$, from (4.18), combining (4.22) and (4.24) yields

$$\|Q_L\|_{k,S} \le CN^{2k} \|Q_L\|_{0,S} \le CL^k \|Q_L\|_{0,S} \tag{4.25}$$

This is the desired result (4.25), and it completes the proof of Theorem 4.1. ∎

For polynomials, the inverse estimates (4 17) are proved in [1,3], and the similar inverse estimates (4.20) hold for RBF, based on the approximation properties. Theorem 4.1 enables us to extend the combined methods and collocation methods in [2,3] to those using the RBF. Similar arguments may be used for the collocation methods and their combinations using the Sinc functions [20], or the fundamental functions.

REMARK 4.1. Equation (4.14) implies the approximation $f \approx Q_L$, having small relative errors. Note that the bound of (4.15) can also be derived from [10]. In fact, we have from [10, Theorem 5.3],

$$\|f - Q_L\|_{k,S} \le Ch^{m-k} \|f\|_{m,S} , \qquad m \ge k, \tag{4.26}$$

where $h = \delta$. Choosing $m = 2k$, we obtain from (4 17),

$$\|f_N - Q_L\|_{k,S} \le Ch^k \|f_N\|_{2k,S} \le Ch^k N^{4k} \|f_N\|_{0,S} \le C_1 \|f_N\|_{0,S} , \tag{4.27}$$

provided that h can be chosen so small that $h^k N^{4k} \le C_1$. This also leads to assumption (4.15). For $c \ge 1$, we may choose small h, such that $(h/c)^k N^{4k} \le C$.

REMARK 4.2. By following [3], the solution domain can be extended to a polygon. Let us briefly describe the arguments in [3]. A polygon can be decomposed to finite parallelograms with overlaps. A parallelogram may be transformed to a rectangle by linear transformations. Then, equation (4.20) also holds for the polygonal domain

5. NUMERICAL EXPERIMENTS

First, we carry out a computational procedure for smooth problems to support the theoretical analysis in Section 3. In Sections 5.2 and 5.3, we carry out two methods for solving Motz's problem. Numerical experiments of combinations of FEM and RBCM will be reported elsewhere.

5.1. Radial Basis Collocation Methods with Various Boundary Conditions

Consider Poisson's equation,

$$-\Delta u = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f(x, y), \qquad \text{in } S, \tag{5.1}$$

where $S = \{(x, y) \mid 0 < x < 1,\ 0 < y < 1\}$, with the mixed type of different boundary conditions,

$$\begin{gathered} u\mid_{x=0} = 0, \qquad u\mid_{y=0} = 0, \\ u_\nu \mid_{x=1} = g_N, \\ u_\nu + \alpha u \mid_{y=1} = g_R, \end{gathered} \tag{5.2}$$

where $\alpha > 0$, (e.g., $\alpha = 2$). The exact solution is chosen to be exactly the same as in [11]

$$u = \sin\left(\frac{\pi x}{6}\right) \sin\left(\frac{7\pi x}{4}\right) \sin\left(\frac{3\pi y}{4}\right) \sin\left(\frac{5\pi y}{4}\right). \tag{5.3}$$

Then, the functions f, g_N and g_R are given explicitly.

The admissible functions are chosen as follows,

$$v = \sum_{i=1}^{L} \tilde{a}_i g_i(x, y), \qquad \text{in } S, \tag{5.4}$$

where $\tilde{a}_i$ are unknown coefficients to be determined, and $g_i(x, y)$ are the RBF. First, we choose the inverse multiquadric radial basis functions (IMQRB)

$$g_i(x, y) = \frac{1}{\sqrt{r_i^2 + c^2}}, \tag{5.5}$$

where c is a shape parameter constant, $r_i = \sqrt{(x - x_i)^2 + (y - y_i)^2}$, and (x_i, y_i) are the source points which should also be chosen as the collocation nodes. Suitable additional functions may be added into (5.4), such as some polynomials or singular functions if necessary. Next, we may choose the Gaussian radial basis functions (GRB),

$$g_i(x, y) = \exp\left(-\frac{r_i^2}{c^2}\right). \tag{5.6}$$

In Section 2, we address that the number of collocation nodes may be larger than the number of terms of RBF, also see Remark 3.1. Let L be the number of RBF, by choosing $L = N^2$. We use the collocation equations (3.17)–(3.19) on the uniform interior and boundary nodes. The distribution of source points in this paper is chosen to be uniform for easy test, but it may, of course, be chosen rather arbitrarily. Then, $\delta = O(1/N)$, where N denotes the number of source points in one direction. The error norms, by using IMQRB with shape parameter $c = 2.0$, are listed in Table 1, where the collocation nodes are also chosen to be uniform and the total number is 196. From Table 1, we can see the following asymptotic relations,

$$\|u - v\|_{0,\infty,S} = O\left((0.22)^N\right), \tag{5.7}$$

$$\|u - v\|_{0,S} = O\left((0.24)^N\right), \tag{5.8}$$

$$\|u - v\|_{1,S} = O\left((0.20)^N\right). \tag{5.9}$$

Table 1 The error norms and condition number by the IMQRB collocation method with parameter $c = 2.0$

$L = N^2$	7^2	9^2	11^2	13^2
$\lVert u - v \rVert_{0,\infty,S}$	9.30 (−3)	5.92 (−5)	4.32 (−6)	1.10 (−6)
$\lVert u - v \rVert_{0,S}$	1.60 (−3)	1.51 (−5)	8.19 (−7)	3.10 (−7)
$\lVert u - v \rVert_{1,S}$	2.28 (−2)	1.32 (−4)	5.22 (−6)	1.67 (−6)
Cond (A)	1.02 (7)	2.28 (8)	1.21 (9)	2.01 (9)

Equations (5.7)–(5.9) indicate that the numerical solutions have the exponential convergence rates. The profiles of exact solutions are shown in Figure 1, and the approximate solutions and errors in Figures 2 and 3. In Table 1, the errors in the Sobolev norm and the infinite norm can be achieved in the order of 10^{-6}, when the number of RBF is given by $L = 11^2$. Moreover, the shape parameter c of RBF can also be chosen larger, e.g., $c = 2.5$ or 3.0. It seems that those numerical results are better than those in [11].

5.2. Adding Method of Singular Functions

Consider Motz's problem,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \text{in } S, \tag{5.10}$$

where $S = \{(x, y) \mid -1 < x < 1,\ 0 < y < 1\}$, with the mixed type of Dirichlet-Neumann conditions,

$$\begin{aligned}
u_x &= 0, && \text{on } x = -1 && \wedge && 0 \le y \le 1, \\
u &= 500, && \text{on } x = 1 && \wedge && 0 \le y \le 1, \\
u_y &= 0, && \text{on } y = 1 && \wedge && -1 \le x \le 1, \\
u &= 0, && \text{on } y = 0 && \wedge && -1 \le x < 0, \\
u_y &= 0, && \text{on } y = 0 && \wedge && 0 < x \le 1.
\end{aligned} \tag{5.11}$$

The origin $(0, 0)$ is a singular point, since the solution behaviour $u = O(r^{1/2})$ as $r \to 0$ due to the intersection of the Dirichlet and Neumann conditions. The exact solution is given in [1], and the leading six coefficients are

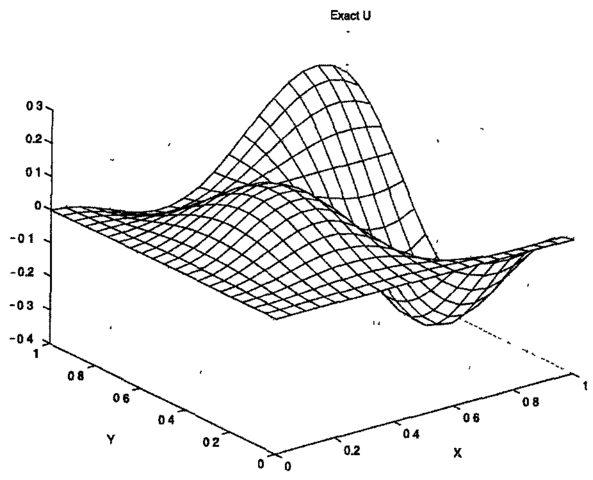
$$\begin{aligned}
d_0 &= 401.1624, \\
d_1 &= 87.6559, \\
d_2 &= 17.2379, \\
d_3 &= -8.0712, \\
d_4 &= 1.44027, \\
d_5 &= 0.33105.
\end{aligned}$$

The profiles of the exact solution are shown in Figure 4.

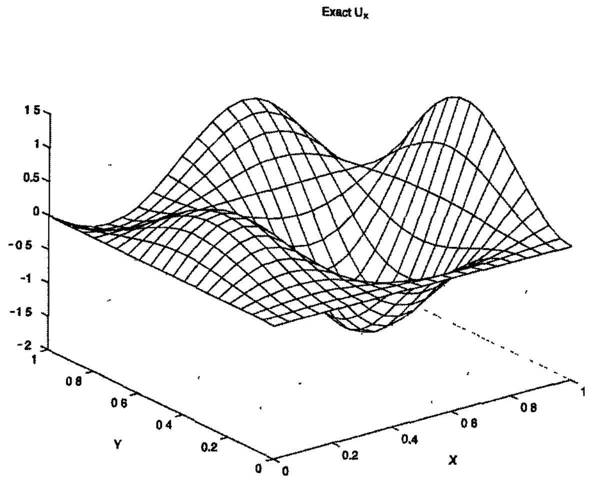
Because of singularity, some singular functions may be added into the RBF. The admissible functions are chosen as

$$v = \sum_{i=1}^{L} \tilde{a}_i g_i(x, y) + \sum_{n=0}^{M} \tilde{d}_n \varphi_n(r, \theta), \qquad \text{in } S, \tag{5.12}$$

where $\tilde{a}_i$ and $\tilde{d}_n$ are unknown coefficients to be determined, and $g_i(x, y)$ are the RBF. In (5.12), $\varphi_n(r, \theta) = r^{n+1} \cos(n + 1/2)\theta$, and equation (2.15) is not used to impose the admissible functions in computation. We choose IMQRB and GRB. Let the distribution of source points also be



(a)



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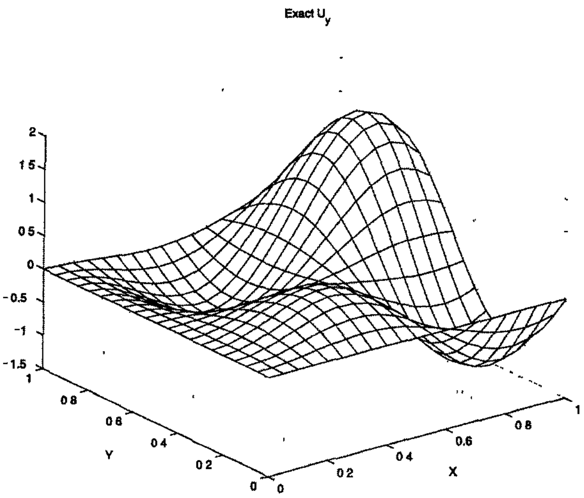


(c)

Figure 1. The exact solution and derivatives for smoothness problem

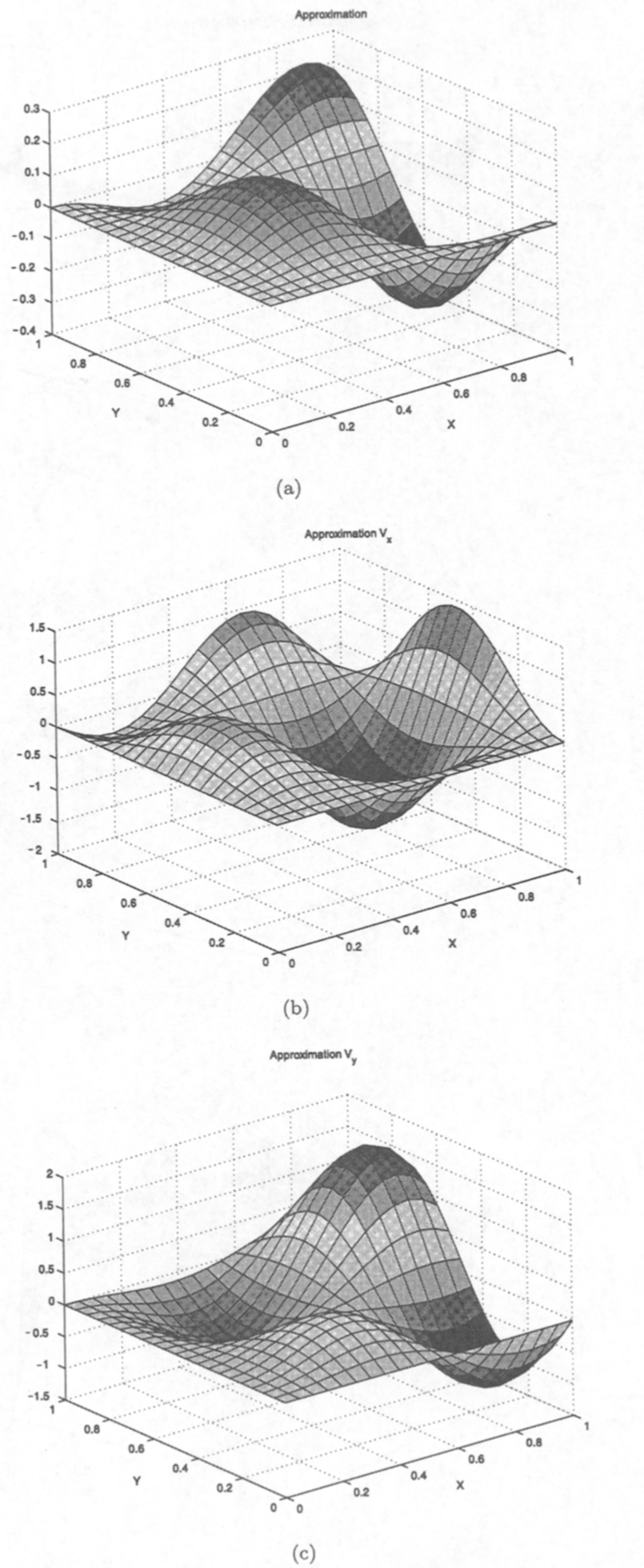


Figure 2 The profiles of solution and derivatives by IMQRB collocation method with $L = 11^2$ and $c = 2.0$.

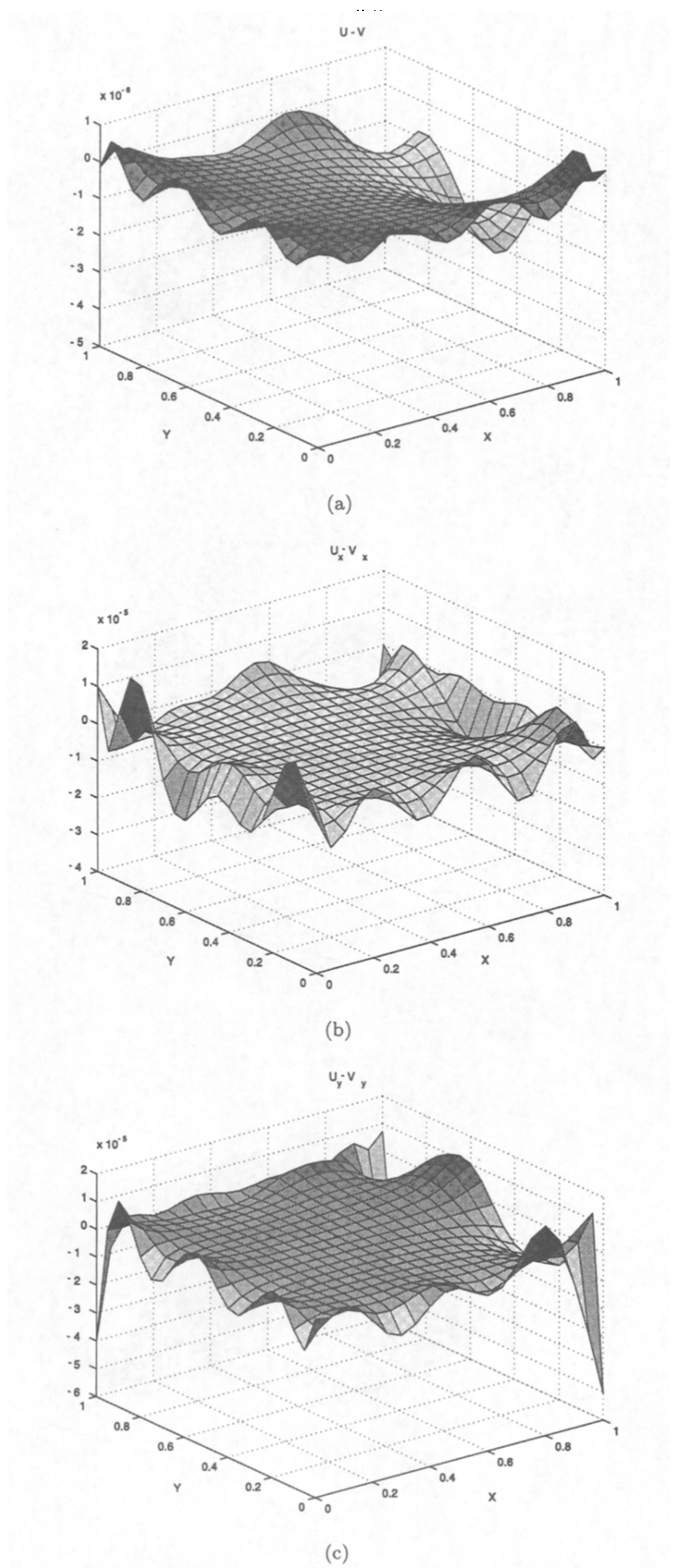
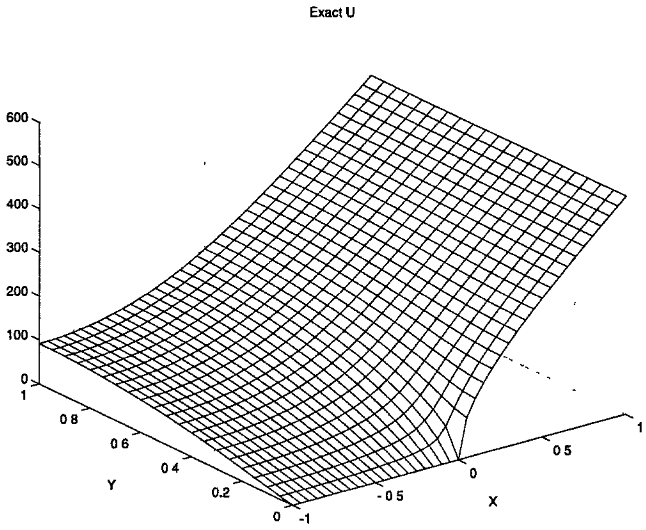


Figure 3. The profiles of the errors by IMQRB collocation method with $L = 11^2$ and $c = 2\ 0$



(a)

Figure 4. The exact solution and derivatives for Motz's problem

uniform in computation, and use collocation equations (3.17)–(3.19) on uniform interior and boundary collocation nodes. The error norms are listed in Tables 2 and 3, for IMQRB ($c = 2.0$) and GRB ($c = 2.0$), respectively. L denotes the number of RBF, $L = N^2$, and M denotes the number of singular functions. The coupling techniques for L and M may refer to [1,2]. From Table 2, we can see the following asymptotic relations,

$$\|u - v\|_{0,\infty,S} = O\left((0.28)^N\right), \tag{5.13}$$

$$\|u - v\|_{0,S} = O\left((0.29)^N\right), \tag{5.14}$$

$$\|u - v\|_{1,S} = O\left((0.33)^N\right). \tag{5.15}$$

Also, from Table 3, we can see for GRB,

$$\|u - v\|_{0,\infty,S} = O\left((0.30)^N\right), \tag{5.16}$$

$$\|u - v\|_{0,S} = O\left((0.31)^N\right), \tag{5.17}$$

$$\|u - v\|_{1,S} = O\left((0.36)^N\right). \tag{5.18}$$

Equations (5.13)–(5.18) indicate that the numerical solutions obtained also have the exponential convergence rates, which verify the error bounds obtained. The profiles of approximate solutions and their errors for Motz's problem are shown in Figures 5 and 6.

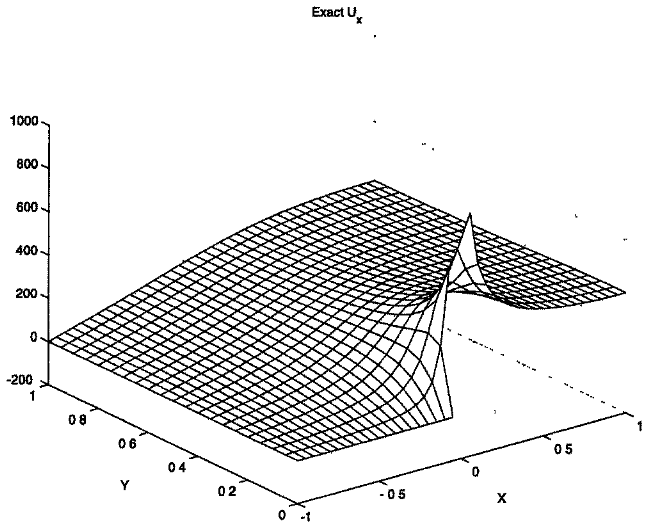
5.3. Subtracting Method of Singular Functions

We also consider Motz's problem. First, choose purely the radial basis functions (RBF),

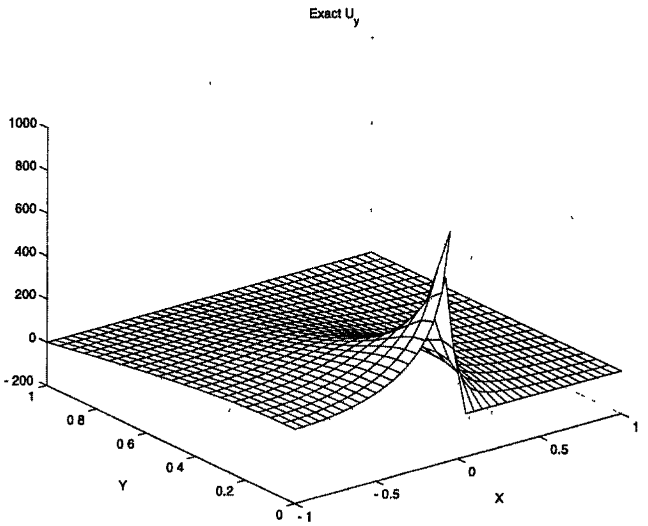
$$\bar{u} = \sum_{i=1}^{L} a_i g_i(x, y). \tag{5.19}$$

The Motz's solutions are known as

$$u(r, \theta) = \sum_{i=0}^{\infty} d_i r^{i+1/2} \cos\left(i + \frac{1}{2}\right)\theta, \tag{5.20}$$



(b)



(c)

where the coefficients can also be obtained from

$$d_i = \frac{2}{\pi} r^{-(i+1/2)} \int_0^{\pi} u(r,\theta) \cos\left(i + \frac{1}{2}\right) \theta \, d\theta. \tag{5.21}$$

Usually, the solutions from the collocation method using (5.19) are poor only near the origin due to the singularity. Hence, choosing $r \geq 1/2$, we may evaluate the approximate coefficients $\tilde{d}_i$ from (5.21) very well, and obtain the singular solutions,

$$\bar{w} = \sum_{i=0}^{M} \tilde{d}_i r^{i+1/2} \cos\left(i + \frac{1}{2}\right) \theta. \tag{5.22}$$

We may subtract the singular part (5.22) from u, and then, obtain a rather smooth problem for

Table 2 The error norms and condition number by the IMQRB collocation method adding singular functions with parameter $c = 2.0$.

$L = N^2,\ M$	$4^2,\ 2$	$6^2,\ 3$	$8^2,\ 4$	$10^2,\ 5$
$\|u - v\|_{0,\infty,S}$	11 34	1.07	7 46 (−2)	5.50 (−3)
$\|u - v\|_{0,S}$	3.00	2 85 (−1)	1 27 (−2)	1.70 (−3)
$\|u - v\|_{1,S}$	12.0	1 68	1 03 (−1)	1.45 (−2)
$\tilde{d}_0$	423 8219	403 9276	401.0821	401 1446
$\tilde{d}_1$	95 9485	111 1686	89 9612	87.7384
$\tilde{d}_2$	12 7307	26 5319	8 6550	16 5567
$\tilde{d}_3$	/	−12.2469	−10 7710	−3.4625
$\tilde{d}_4$	/	/	1.8867	−3.1864
$\tilde{d}_5$	/	/	/	−0 8645
Cond (A)	2 58 (3)	8.60 (5)	5 38 (8)	3 21 (9)

Table 3 The error norms and condition number by the GRB collocation method adding singular functions with parameter $c = 2\ 0$.

$L = N^2,\ M$	$4^2,\ 2$	$6^2,\ 3$	$8^2,\ 4$	$10^2,\ 5$
$\|u - v\|_{0,\infty,S}$	8 07	7 32 (−1)	9.40 (−2)	5 90 (−3)
$\|u - v\|_{0,S}$	1 86	1 95 (−1)	1 39 (−2)	1 60 (−3)
$\|u - v\|_{1,S}$	8 13	1 07	1 45 (−1)	1.78 (−2)
$\tilde{d}_0$	416 4048	403 6369	401.0648	401 1865
$\tilde{d}_1$	72.9120	103.3334	88.9340	87 3105
$\tilde{d}_2$	16 4691	45 5082	24.7180	17.2233
$\tilde{d}_3$	/	−19.0671	−11 6812	−13.7762
$\tilde{d}_4$	/	/	−1 7538	7 8214
$\tilde{d}_5$	/	/	/	−0.4210
Cond. (A)	7.16 (3)	2.73 (8)	1 36 (9)	5.79 (9)

$\bar{u} = u - \bar{w}$, with the following equations,

$$\Delta \bar{u} = 0, \qquad \text{in } S, \tag{5.23}$$

$$\begin{aligned}
\bar{u}_x &= -\bar{w}_x, && \text{on } x = -1 \ \wedge \ 0 \le y \le 1, \\
\bar{u} &= 500 - \bar{w}, && \text{on } x = 1 \ \wedge \ 0 \le y \le 1, \\
\bar{u}_y &= -\bar{w}_y, && \text{on } y = 1 \ \wedge \ -1 \le x \le 1, \\
\bar{u} &= 0, && \text{on } y = 0 \ \wedge \ -1 \le x < 0, \\
\bar{u}_y &= 0, && \text{on } y = 0 \ \wedge \ 0 < x \le 1.
\end{aligned} \tag{5.24}$$

Again, we use the radial basis collocation method in Section 3 to seek the rather smooth problem (5.23),(5 24). We add singular part $\bar{w}$ to the smooth part $\bar{u}$ to obtain a better approximation to u. Repeat the evaluation (5.21) of coefficients d_i, and then, subtract $\bar{w}$ of (5.22) again. The above iteration repeats until a convergent solution is obtained.

The error norms are listed in Tables 4 and 5 for IMQRB ($c = 2.0$) and GRB ($c = 2.0$), respectively. We choose $r = 1$ to evaluate the approximate coefficients $\tilde{d}_n$, $n = 0, 1, \ldots, 5$. The termination of iterations occurs when the absolute error becomes less than 10^{-6}, and the number of iteration needed is about 10. From Table 4, we can also observe the exponential convergence

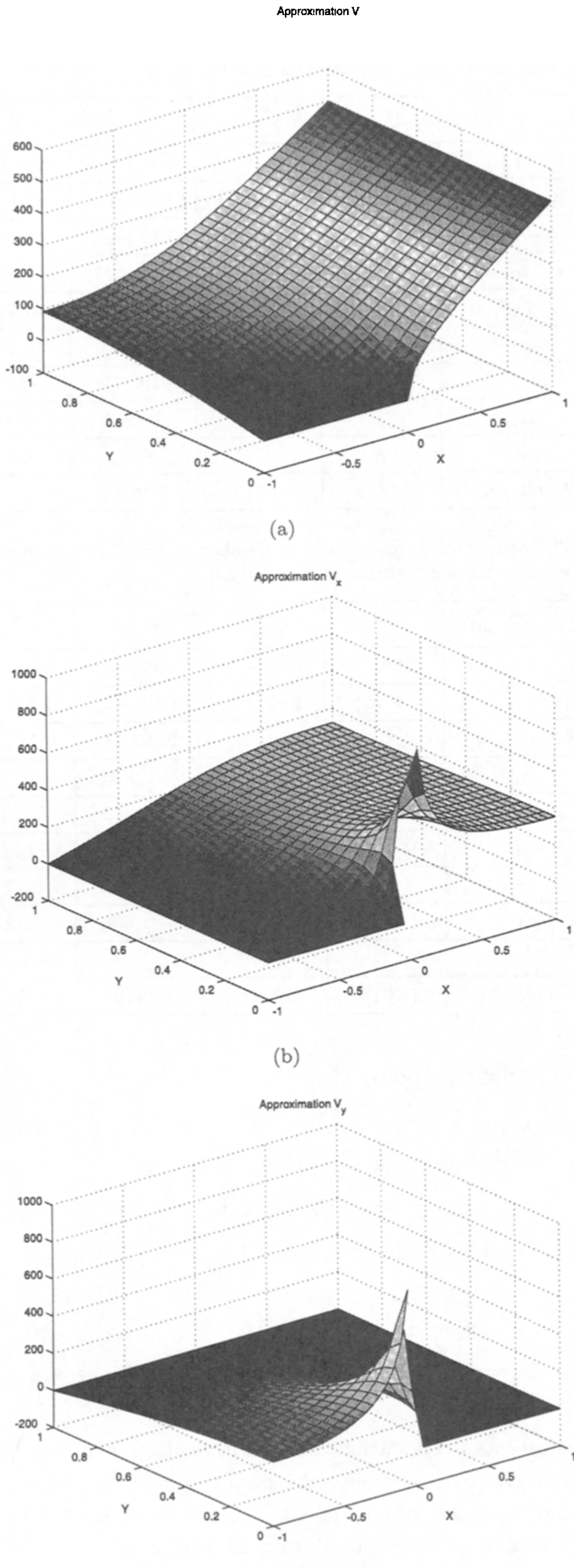
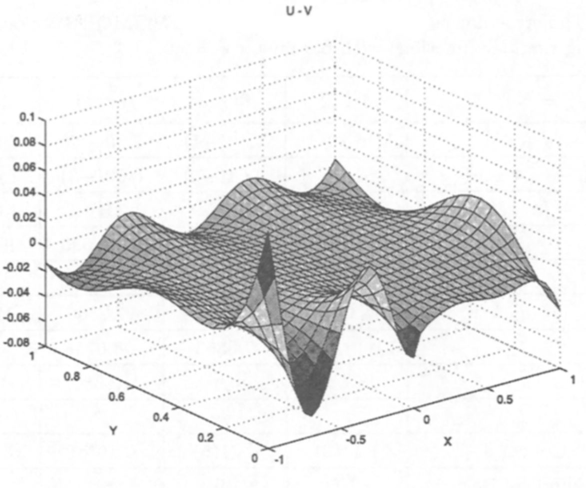
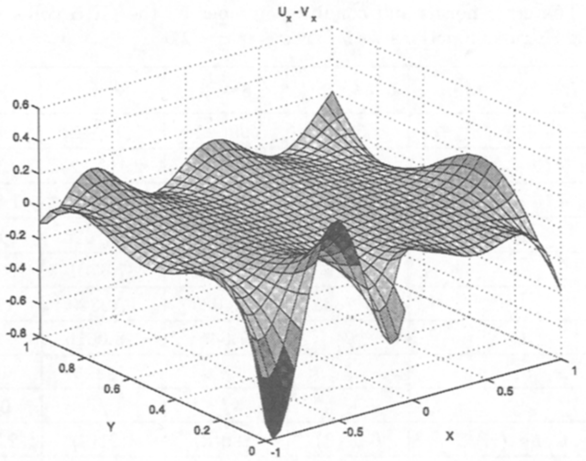


Figure 5 The profiles of solution and derivatives by GRB collocation method adding singular functions with $L = 8^2, M = 4$, and $c = 2\ 0$.



(a)



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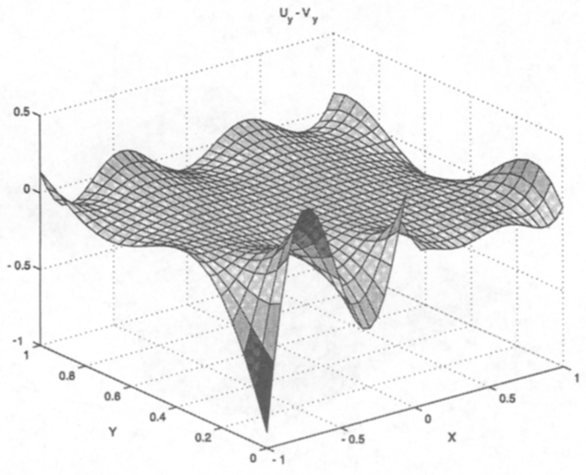


(c)

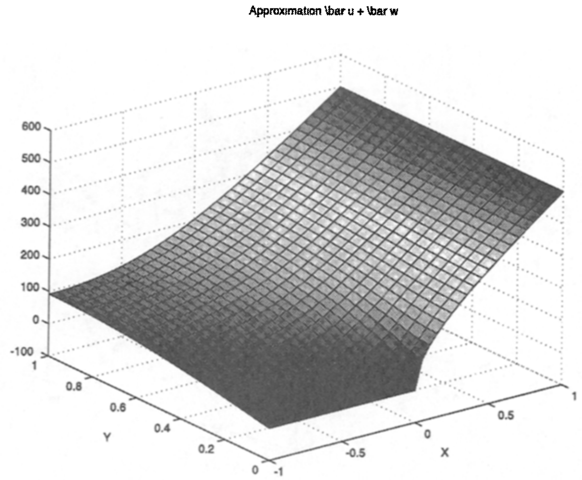
Figure 6 The profiles of errors by GRB collocation method adding singular functions with $L = 8^2, M = 4$, and $c = 2.0$.

Table 4. The error norms and condition number by the IMQRB collocation method subtracting singular functions with parameter $c = 2\ 0$

$L = N^2,\ M$	$4^2,\ 2$	$6^2,\ 3$	$8^2,\ 4$	$10^2,\ 5$
$\|u - (\bar{u} + \bar{w})\|_{0,\infty,S}$	6 99	6.10(−1)	7 40(−2)	7 00(−3)
$\|u - (\bar{u} + \bar{w})\|_{0,S}$	2.24	2.64(−1)	8 90(−3)	1 40(−3)
$\|u - (\bar{u} + \bar{w})\|_{1,S}$	7.82	1.45	1.11(−1)	1 63(−2)
$\tilde{d}_0$	399.8414	401 0809	401.1633	401 1627
$\tilde{d}_1$	86 0459	87.6951	87.6526	87.6556
$\tilde{d}_2$	18.0743	17.0752	17.2360	17 2378
$\tilde{d}_3$	/	7 8863	−8 0718	−8 0714
$\tilde{d}_4$	/	/	1.4386	1 4403
$\tilde{d}_5$	/	/	/	0.3310
Cond. (A)	1.14(3)	4.21(5)	6.16(8)	3.21(9)
Num of iter.	10	10	9	9

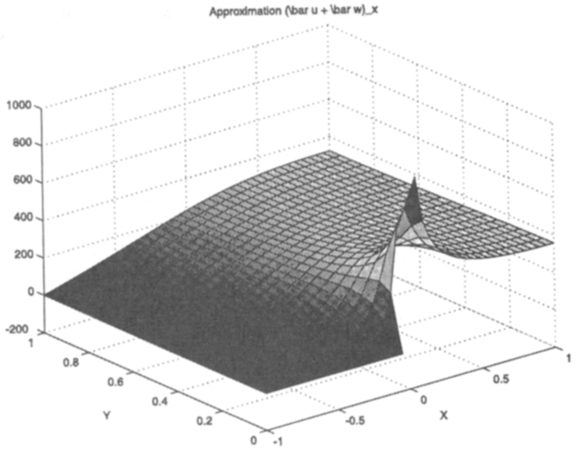
Table 5 The error norms and condition number by the GRB collocation method subtracting singular functions with parameter $c = 2\ 0$.

$L = N^2,\ M$	$4^2,\ 2$	$6^2,\ 3$	$8^2,\ 4$	$10^2,\ 5$
$\|u - (\bar{u} + \bar{w})\|_{0,\infty,S}$	5 24	5 99(−1)	1.01(−1)	1.18(−2)
$\|u - (\bar{u} + \bar{w})\|_{0,S}$	1 62	2.69(−1)	2 10(−2)	2 10(−3)
$\|u - (\bar{u} + \bar{w})\|_{1,S}$	6.14	1 48	1.64(−1)	1 87(−2)
$\tilde{d}_0$	399.7470	401 0842	401 1625	401.1617
$\tilde{d}_1$	87 2125	87 7272	87 6501	87.6571
$\tilde{d}_2$	17.7888	17 1083	17.2329	17.2373
$\tilde{d}_3$	/	−7.8454	−8.0616	−8.0708
$\tilde{d}_4$	/	/	1.4393	1.4403
$\tilde{d}_5$	/	/	/	0.3310
Cond. (A)	6 06(3)	1 89(8)	1.01(9)	2 70(10)
Num of iter.	11	10	10	9



(a).

Figure 7 The profiles of solution and derivatives by GRB collocation method subtracting singular functions with $L = 8^2, M = 4$, and $c = 2.0$



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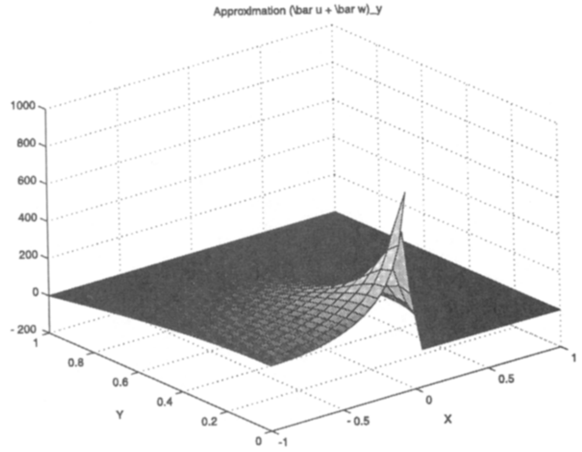
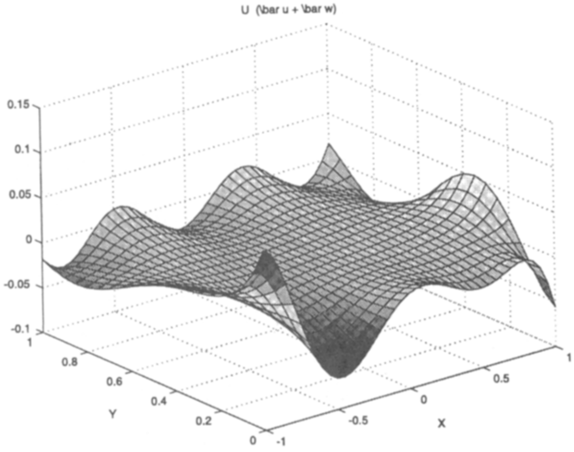


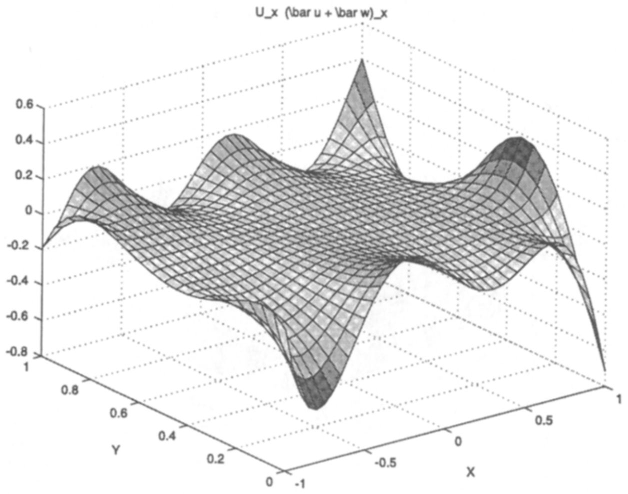
(c).

Figure 7. (cont.)



(a).

Figure 8. The profiles of the errors by GRB collocation method subtracting singular functions with $L = 8^2, M = 4$, and $c = 2\ 0$



(b).

U_y (\bar u + \bar w)_y

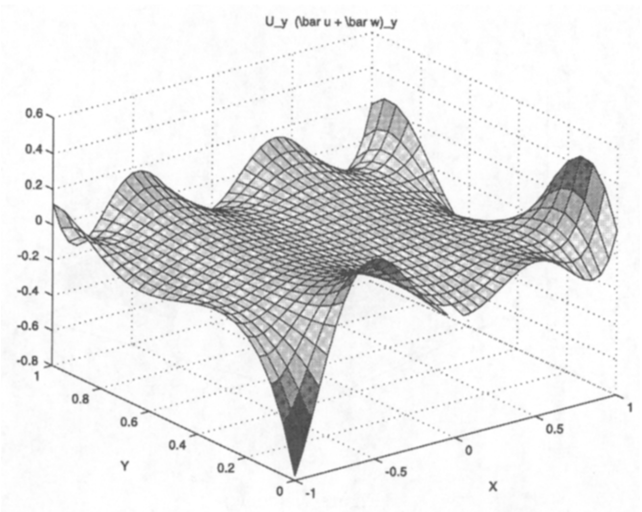


Figure 8 (cont)

rates,

$$\|u - (\bar{u} + \bar{w})\|_{0,\infty,S} = O\left((0.34)^N\right), \tag{5.25}$$

$$\|u - (\bar{u} + \bar{w})\|_{0,S} = O\left((0.30)^N\right), \tag{5.26}$$

$$\|u - (\bar{u} + \bar{w})\|_{1,S} = O\left((0.33)^N\right). \tag{5.27}$$

Also, from Table 5,

$$\|u - (\bar{u} + \bar{w})\|_{0,\infty,S} = O\left((0.39)^N\right), \tag{5.28}$$

$$\|u - (\bar{u} + \bar{w})\|_{0,S} = O\left((0.34)^N\right), \tag{5.29}$$

$$\|u - (\bar{u} + \bar{w})\|_{1,S} = O\left((0.38)^N\right). \tag{5.30}$$

The exponential convergence rates, equations (5.25)–(5.30), also support the theoretical analysis made. The profiles of the approximate solutions and their errors are shown in Figures 7 and 8.

Table 6 The error norms and condition number by the IMQRB collocation method adding singular functions with $L = 8^2$ and $M = 4$

c	$\|u-v\|_{0,\infty,S}$	$\|u-v\|_{0,S}$	$\|u-v\|_{1,S}$	Cond (A)
1 0	6 71(−2)	1.96(−2)	1.17(−1)	2.34(6)
1 2	7 93(−2)	2.16(−2)	1 39(−1)	5 56(6)
1 4	9 36(−2)	2 12(−2)	1 44(−1)	1 66(7)
1.6	9 36(−2)	1 94(−2)	1.37(−1)	5.49(7)
1 8	8 59(−2)	1.63(−2)	1.21(−1)	1 94(8)
2 0	7 46(−2)	1.27(−2)	1 03(−1)	1 70(9)
2 2	6 29(−2)	9 70(−3)	8 74(−2)	8 27(8)
2 4	5 25(−2)	7 60(−3)	7 60(−2)	1 83(9)
2 6	4 38(−2)	6 10(−3)	6.81(−2)	1 92(9)
2 8	3 65(−2)	5 20(−3)	6 28(−2)	3.35(9)
3 0	3 30(−2)	4 80(−3)	5 96(−2)	2 01(9)

Table 7 The error norms and condition number by the IMQRB collocation method subtracting singular functions with $L = 8^2$ and $M = 4$

c	$\|u-(\bar{u}+\bar{w})\|_{0,\infty,S}$	$\|u-(\bar{u}+\bar{w})\|_{0,S}$	$\|u-(\bar{u}+\bar{w})\|_{1,S}$	Cond. (A)
1.0	5 28(−2)	8.70(−3)	9 19(−2)	1 35(6)
1 2	8 02(−2)	9 80(−3)	1.14(−1)	2 97(6)
1 4	9 78(−2)	1 03(−2)	1.27(−1)	7.81(6)
1 6	9.63(−2)	1 01(−2)	1.27(−1)	2 32(7)
1.8	8 66(−2)	9 60(−3)	1 20(−1)	7 35(7)
2 0	7.40(−2)	8 90(−3)	1.11(−1)	1 95(8)
2 2	6.16(−2)	8 20(−3)	1 00(−1)	3.18(8)
2.4	5.06(−2)	7 50(−3)	9 06(−2)	4 75(8)
2 6	4 13(−2)	6 80(−3)	8 20(−2)	5 59(8)
2 8	3 36(−2)	6.20(−3)	7 48(−2)	9.11(8)
3.0	2.73(−2)	5.70(−3)	6 89(−2)	1.11(9)

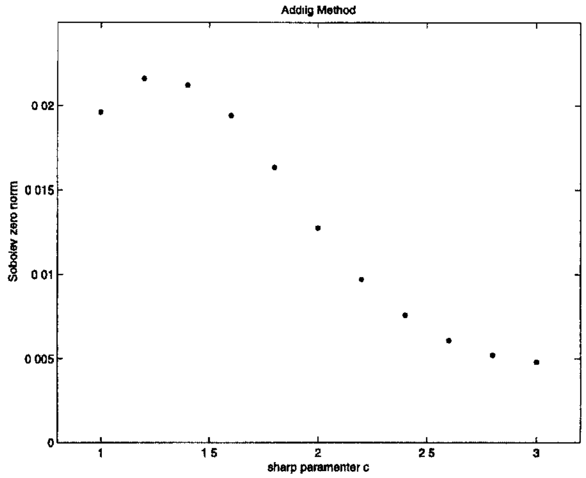
The adding method of singular solutions was reported in [21], and the subtracting method of singular solutions in [22]. (See also [1].)

5.4. Comparisons and Conclusions

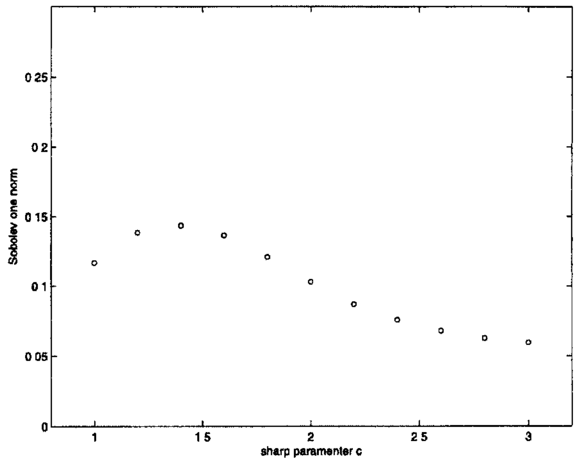
1. To solve Poisson's problem, this paper provides the theoretical framework of RBCM and its combinations with other methods to cover various numerical methods of RBF. This paper is also an important development of RBF from the approximation theory of smooth functions to the solutions of partial differential equations. The RBCM is a fairly new and efficient tool for solving PDEs, in which the solutions in terms of RBF and singular functions are added if necessary.

2. From Table 1, we can observe the exponential convergence rates for smooth problems, which may be competitive to orthogonal polynomials. On the other hand, from Tables 2–5, we can find that there also exist the exponential convergence rates for singularity problems, when applying some singular functions. From these tables, the following asymptotic relations are also observed,

$$\|u-v\|_{k,S} = O\left(\lambda^N\right) = O\left(\lambda^{\sqrt{L}}\right), \qquad k = 0, 1,$$



(a)



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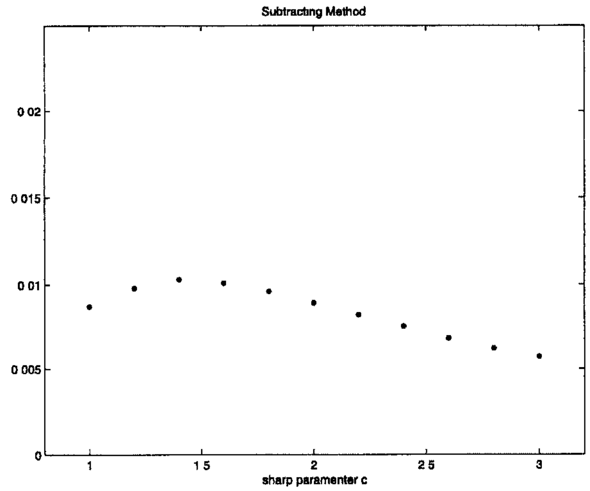
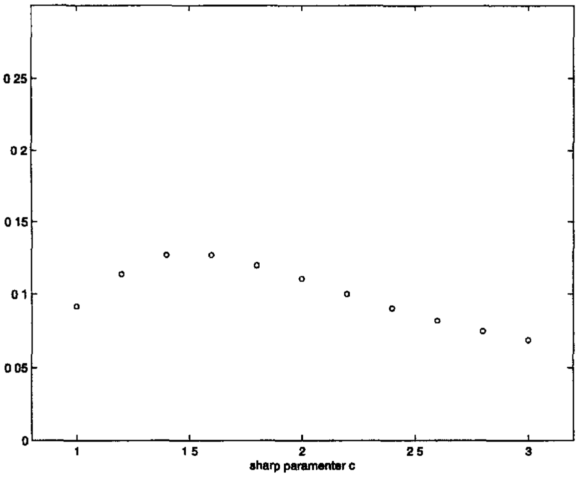


(c)

Figure 9 The solution errors versus parameter c, with $L = 8^2$ and $M = 4$



(d)

Figure 9 (cont)

where $0 < \lambda < 1$.

For Motz's problem, we find that the approximate solutions by RBCM adding method of singular solutions in Section 5.2 have much better convergence rates than those by RBCM subtracting method for the singular solutions in Section 5.3. On the contrary, the leading coefficients of singular functions, $\tilde{d}_n$, $n = 0, 1, \ldots, 5$, obtained by RBCM subtracting method are more accurate than those by RBCM adding method. We also observe that the IMQRB is superior to the GRB, in both accuracy and stability.

3. From the numerical results, we see that the RBF have large condition number which implies high instability. That is the drawback of the radial basis collocation method. In practical computation, since only a few terms of RBF are needed, such drawback is not serious. In spite of this drawback, the RBCM is still a competitive method for PDEs due to high accuracy and a very low computational cost.

4. From Tables 6 and 7 and Figure 9, we can see that there exists convergence by increasing parameter c from 1.0 to 3.0. The following asymptotic relations are also observed,

$$\|u - v\|_{k,S} = O\left(\lambda^c\right), \qquad k = 0, 1,$$

where $0 < \lambda < 1$.

5. All of numerical experiments indicate that the RBCM have exponential convergence rates

$$\|u - v\|_{k,S} = O\left(\lambda^{c/\delta}\right) \approx O\left(\lambda^{cN}\right) \approx O\left(\lambda^{c\sqrt{L}}\right), \qquad k = 0, 1,$$

where $0 < \lambda < 1$. From the viewpoint of accuracy, the errors tend to zero as $c/\delta \to \infty$ (i.e., $L \to \infty$ and $c \to \infty$). However, from the viewpoint of stability, we can not increase L too much due to large condition number. Also, we cannot increase c too much due to the flatness of RBF, causing the ill-conditioned F in (3.20). It seems that accuracy and instability are twins. Such a statement is also called the *uncertainty principle* in [23]. Either one goes for a small error and gets a bad sensitivity, or one wants a stable algorithm and has to take a comparably large error. In real computation, we should keep some balance between them. In summary, Theorem 3.1 and the numerical examples display that the errors of the solutions of Poisson's equation by RBCM have the same exponential convergence rates as those of surface fitting of RBF.

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