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Short Communication

Verification of Reduced Convergence Rates

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Abstract

In this short article, we recalculate the numerical example in Křížek and Neittaanmäki (1987) for the Poisson solution $u = x^\sigma(1-x)\sin \pi y$ in the unit square S as $\sigma = \frac{7}{4}$. By the finite difference method, an error analysis for such a problem is given from our previous study by $\overline{\|\epsilon\|}_1 = C_1 h^2 + C_2 h^{\frac{5}{4}}$, where h is the meshspacing of the uniform square grids used, and C_1 and C_2 are two positive constants. Let $\epsilon = u - u_h$, where u_h is the finite difference solution, and $\overline{\|\epsilon\|}_1$ is the discrete H^1 norm. Several techniques are employed to confirm the reduced rate $O(h^{\frac{5}{4}})$ of convergence, and to give the constants, $C_1 = 0.09034$ and $C_2 = 0.002275$ for a stripe domain. The better performance for $\sigma = \frac{7}{4}$ arises from the fact that the constant C_1 is much large than C_2, and the h in computation is not small enough.

AMS(MOS) Subject Classifications: 65N10, 65N30.

Keywords: Numerical verification, reduced convergence rates, superconvergence, singularity, Poisson equation.

Krizek and Neittaanmaki in [1] considered the Poisson equation on a unit square domain $S = (0,1) \times (0,1)$, and choose the solution

$$u(x,y) = x^\sigma(1-x)\sin \pi y, \quad \text{in} \quad S \ , \tag{1}$$

where $\sigma > \frac{1}{2}$. The recovered gradient technique is used in a post processing, and the superconvergence $O(h^2)$ can be achieved for $u \in H^3(S)$, where h is the maximal boundary length of the triangles used. The numerical results in [1] for $\sigma = 1$ and $\frac{7}{4}$ also showed the best global superconvergence $O(h^2)$. It seems that for $\sigma = \frac{7}{4}$, the post-processing can give good numerical results even when $u \notin H^3(S)$, see [1], p. 228.

For the Poisson equation $-\Delta u = f$ in S and $u = 0$ on ∂S, the finite element method (FEM) is to seek $u_h^E \in V_h$ such that

$$\int\int_S \nabla u_h^E \nabla v = \int\int_S fv, \quad \forall v \in V_h \ , \tag{2}$$

where V_h is a finite collection of piecewise linear functions on S satisfying $v = 0$ on ∂S. Denote $S = \cup_{ij} \triangle_{ij}$, where $\triangle_{ij}$ are small triangles. When $\triangle_{ij}$ are uniform and right triangles, the finite difference equations of five nodes are obtained. On the other hand, the Shortley-Weller difference scheme is also of five nodes, which can be interpreted as the linear or bilinear FEM using special rules of integrations [4]: To seek $u_h \in V_h$ such that

$$\widehat{\iint_S} \nabla u_h \nabla v = \widehat{\iint_S} fv, \quad \forall v \in V_h \ , \tag{3}$$

where $\widehat{\int\int_S}$ is an approximation of $\int\int_S$. Eqs. (2) and (3) lead to the linear algebraic equations

$$\mathbf{A}\vec{x} = \vec{b}, \quad \text{and} \quad \mathbf{A}\vec{x} = \vec{b}^* \ , \tag{4}$$

respectively, where the same matrix $\mathbf{A}$ is positive definite and spares, and $\vec{x}$ is the unknown vector consisting of $(u_h^E)_{ij}$ (or $(u_h)_{ij}$) at the interior modes (i, j). In (4), the vector $\vec{b}$ results from $\int\int_S fv$, and is evaluated in the computation [1] by the seven-point numerical integration formula given in book [2], p.58, which is exact for all quintic polynomials. Of course, we can also choose the Gaussian rules with high order on triangles $\triangle_{ij}$ in Strang and Fix [6]. Hence we may assume that $\int\int_S$ can be evaluated "almost" exactly. Note that only the non-homogeneous terms $\vec{b}$ and $\vec{b}^*$ in (4) are different. Based on the analysis in [5], both the errors from $\widehat{\int\int_S} fv$ and the global errors of u_h have the same order of h, where h is the maximal meshspacing of the rectangular grids. Hence we may reasonably choose the computational schemes in [5] we are familiar with, to investigate numerically the errors behavior for the example of $\sigma = \frac{7}{4}$ in [1].

Since the stiffness matrix in [1] from the linear elements on the uniform and right triangles is just the finite difference scheme of five nodes, and since the Poisson solution (1) is exactly the same as that in our recent paper [5], we will use the Shortley-Weller difference scheme in [5], to recompute the example in [1] and to report some new discoveries. First, we run our program for $\sigma = 2, \frac{7}{4}$ and $\frac{3}{2}$ as the uniform square grids are chosen. When $\sigma = 2$, the solution is highly smooth, and when $\sigma = \frac{7}{4}$ and $\frac{3}{2}$, $u \notin H^3(S)$, where the case of $\sigma = \frac{7}{4}$ was given in [1]. Note that the numerical experiments for $\sigma = \frac{7}{6}, 0.95$ and $\frac{1}{2}$ using a local refinement of difference grids near the axis Y have been reported in [5]. The error norms are defined exactly the same way as in [5], and the results are listed in Tables 1–3. In [5], we define the discrete H^1 norms:

$$\overline{\|v\|}_1^2 = \overline{\|v\|}_{1,S}^2 = \overline{|v|}_{1,S}^2 + \overline{\|v\|}_{0,S}^2 \ , \tag{5}$$

$$\overline{|v|}_1^2 = \overline{|v|}_{1,S}^2 = \sum_{ij} \left[\widehat{\iint}_{\square_{ij}} (\nabla v)^2 ds \right] \ , \tag{6}$$

Table 1. Errors and condition numbers for $\sigma = 2$ and the uniform square grids

N, N_a, N_b	4,5,3	8,10,6	16,20,12	32,40,24
$\overline{\|\epsilon\|}_0$	0.187(-2)	0.423(-3)	0.103(-3)	0.254(-4)
$\overline{\|\epsilon\|}_1$	0.144(-1)	0.362(-2)	0.905(-3)	0.226(-3)
$\max_{ij} \|\epsilon_{ij}\|$	0.559(-2)	0.141(-2)	0.352(-3)	0.882(-4)
$\max_{ij} \|(\epsilon_x)_{i+\frac{1}{2},j}\|$	0.269(-1)	0.703(-2)	0.179(-2)	0.452(-3)
$\max_{ij} \|(\epsilon_y)_{i,j+\frac{1}{2}}\|$	0.193(-1)	0.523(-2)	0.134(-2)	0.336(-3)
Av	0.203(-2)	0.426(-3)	0.972(-4)	0.232(-4)
Av_x	0.154(-1)	0.345(-2)	0.814(-3)	0.198(-3)
Av_y	0.689(-2)	0.159(-2)	0.386(-3)	0.948(-4)
ϵ_1	0.395(-2)	0.538(-3)	0.688(-4)	0.864(-5)
ϵ_2	0.559(-2)	0.995(-3)	0.135(-3)	0.172(-4)
$\max_j \|\epsilon_{1j}\|$	0.995(-3)	0.124(-3)	0.155(-4)	0.194(-5)
$\max_j \|(\epsilon_x)_{\frac{1}{2},j}\|$	0.205(-1)	0.513(-2)	0.128(-2)	0.321(-3)
$Con(\mathbf{A})$	22.0	110	495	0.185(4)

Table 2. Errors and condition numbers for $\sigma = \frac{7}{4}$ and the uniform square grids

N, N_a, N_b	4,5,3	8,10,6	16,20,12	32,40,24
$\overline{\|\epsilon\|}_0$	0.186(-2)	0.416(-3)	0.994(-4)	0.243(-4)
$\overline{\|\epsilon\|}_1$	0.137(-1)	0.346(-2)	0.869(-3)	0.218(-3)
$\max_{ij} \|\epsilon_{ij}\|$	0.486(-2)	0.123(-2)	0.305(-3)	0.748(-4)
$\max_{ij} \|(\epsilon_x)_{i+\frac{1}{2},j}\|$	0.262(-1)	0.705(-2)	0.183(-2)	0.468(-3)
$\max_{ij} \|(\epsilon_y)_{i,j+\frac{1}{2}}\|$	0.186(-1)	0.507(-2)	0.128(-2)	0.318(-3)
Av	0.197(-2)	0.386(-3)	0.848(-4)	0.196(-4)
Av_x	0.147(-1)	0.356(-2)	0.798(-3)	0.200(-3)
Av_y	0.602(-2)	0.136(-2)	0.318(-3)	0.755(-4)
ϵ_1	0.344(-2)	0.472(-3)	0.596(-4)	0.741(-5)
ϵ_2	0.487(-2)	0.872(-3)	0.117(-3)	0.146(-4)
$\max_j \|\epsilon_{1j}\|$	0.583(-3)	0.152(-3)	0.160(-4)	0.194(-5)
$\max_j \|(\epsilon_x)_{\frac{1}{2},j}\|$	0.169(-1)	0.345(-2)	0.317(-3)	0.258(-3)
$Con(\mathbf{A})$	22.0	110	459	0.185(4)

Table 3. Errors and condition numbers for $\sigma = \frac{3}{2}$ and the uniform square grids

N, N_a, N_b	4,5,3	8,10,6	16,20,12	32,40,24
$\overline{\|\epsilon\|}_0$	0.187(-2)	0.446(-3)	0.113(-3)	0.307(-4)
$\overline{\|\epsilon\|}_1$	0.118(-1)	0.305(-2)	0.807(-3)	0.243(-3)
$\max_{ij} \|\epsilon_{ij}\|$	0.350(-2)	0.989(-3)	0.255(-3)	0.664(-4)
$\max_{ij} \|(\epsilon_x)_{i+\frac{1}{2},j}\|$	0.213(-1)	0.609(-2)	0.256(-2)	0.235(-2)
$\max_{ij} \|(\epsilon_y)_{i,j+\frac{1}{2}}\|$	0.151(-1)	0.396(-2)	0.932(-3)	0.203(-3)
Av	0.149(-2)	0.303(-3)	0.736(-4)	0.214(-4)
Av_x	0.118(-1)	0.289(-2)	0.784(-3)	0.222(-3)
Av_y	0.417(-2)	0.899(-3)	0.206(-3)	0.513(-4)
ϵ_1	0.350(-2)	0.518(-3)	0.119(-3)	0.450(-4)
ϵ_2	0.315(-2)	0.873(-3)	0.131(-3)	0.510(-4)
$\max_j \|\epsilon_{1j}\|$	0.343(-3)	0.275(-3)	0.119(-3)	0.450(-4)
$\max_j \|(\epsilon_x)_{\frac{1}{2},j}\|$	0.787(-2)	0.940(-3)	0.256(-2)	0.235(-2)
$Con(\mathbf{A})$	22.0	110	459	0.185(4)

$$\overline{\|v\|}_0^2 = \overline{\|v\|}_{0,S}^2 = \sum_{ij} \left[\widehat{\iint_{\square_{ij}}} v^2 ds \right] , \tag{7}$$

where $\widehat{\iint}_{\square_{ij}} v^2 ds$ is the integration quadrature approximation of $\iint_{\square_{ij}} v^2 ds$, and $\square_{ij} = \{(x,y) \big| x_i \le x \le x_{i+1}, y_j \le y \le y_{j+1}\}$.

Let u_h be the solution of the Shortley-Weller difference approximation for the Poisson equation. Denote $\epsilon_x = u_x - (u_h)_x$ and $\epsilon_y = u_y - (u_h)_y$, where $u_x = \frac{\partial u}{\partial x}$ and $u_y = \frac{\partial u}{\partial y}$. Also define the average norms in [5]:

$$Av = \frac{1}{Num} \sum_{ij} |\epsilon(i,j)|, \quad Av_x = \frac{1}{Num} \sum_{ij} \left| \epsilon_x(i+\frac{1}{2}, j) \right| , \tag{8}$$

$$Av_y = \frac{1}{Num} \sum_{ij} \left| \epsilon_y(i, j+\frac{1}{2}) \right| ,$$

$$\epsilon_1 = \max_{i,j} \{ |\epsilon_{1,j}|, |\epsilon_{2N-1,j}|, |\epsilon_{i,1}|, |\epsilon_{i,N-1}| \}, \quad \epsilon_2 = \max_{i,j} \{ |\epsilon_{2,j}|, |\epsilon_{2N-2,j}|, |\epsilon_{i,2}|, |\epsilon_{i,N-2}| \} ,$$

where $\epsilon(i,j) = \epsilon_{i,j} = \epsilon(ih,,jh)$, h is the meshspacing of the uniform rectangular grids used in the finite difference method, and *Num* is the total number of the errors related. Besides, $Con(\mathbf{A})$ denotes the condition number of the associated matrix $\mathbf{A}$.

Looking at Tables 1–3, the ratios of $\overline{\|\epsilon\|}_1$ in the last two columns are given by

$$\frac{0.905(-3)}{0.226(-3)} = 4.00 = 2^2 = O(h^2), \quad as \quad \sigma = 2, \tag{9}$$

$$\frac{0.869(-3)}{0.218(-3)} = 3.986 = 2^{1.995} = O(h^{2-\delta}), \quad 0 < \delta << 1, \quad as \quad \sigma = \frac{7}{4}, \tag{10}$$

$$\frac{0.807(-3)}{0.243(-3)} = 3.32 = 2^{1.73} = O(h^{\frac{5}{3}}), \quad as \quad \sigma = \frac{3}{2}.$$

Numerically, we can see

$$\overline{\|\epsilon\|}_1, Av_x = O(h^2), \quad as \quad \sigma = 2, \tag{11}$$

$$\overline{\|\epsilon\|}_1, Av_x = O(h^{2-\delta}), \quad 0 < \delta << 1, \quad as \quad \sigma = \frac{7}{4}, \tag{12}$$

$$\overline{\|\epsilon\|}_1, Av_x = O(h^{\frac{5}{3}}), \quad as \quad \sigma = \frac{3}{2}. \tag{13}$$

Equations (12) and (13) are not coincident with the theoretical predictions $O(h^{\sigma-\frac{1}{2}})$ for the uniform rectangular grids in [5]:

$$\overline{\|\epsilon\|}_1 = O(h^{\frac{5}{4}}), \quad as \quad \sigma = \frac{7}{4}, \tag{14}$$

$$\overline{\|\epsilon\|}_1 = O(h), \quad as \quad \sigma = \frac{3}{2}. \tag{15}$$

Note that Eq. (12) is very similar to the results for Example 5.2 in [1].

From Tables 1–3, we may see numerically that the real deficit of the convergence order occurs for $\sigma = \frac{3}{2}$ but not for $\sigma = \frac{7}{4}$, see (12) and (13) which are so different because $u \in H^2(S)$ stands for $\sigma = \frac{3}{2}$ but not for $\sigma = \frac{7}{4}$. All data in Tables 1–4 have been obtained by Fortran programs in double precision. Note that the norm equivalence between $\overline{\|\epsilon\|}_1$ and $\|u_I - u_h\|_{1,S}$ can be found in [4], p. 395, where u_I and u_h are the interpolant solution of the true solution u and the FEM (or FDM) solution u_h, respectively. Hence, although Tables 1–3 are carried out for the superconvergence of $\overline{\|\epsilon\|}_1$ and Av_x (the average nodal derivatives), the conclusions may be drawn similarly for the global superconvergence in [1].

From [5] the reduced convergence rates by the Shortley-Weller difference approximation for the Poisson solution (1) can be obtained as

$$\overline{\|\epsilon\|}_1 \le C_1 h^2 + C_2 h^{\sigma - \frac{1}{2}}, \quad as \quad \frac{1}{2} < \sigma < 2 \wedge \sigma \ne 1, \tag{16}$$

where C_1 and C_2 are positive constants. When $C_1 >> C_2$ and h is not small, the computed results may not show the desired order $O(h^{\frac{5}{4}})$. The order $O(h^{\frac{5}{4}})$ can only be observed numerically, if the special model on a stripe domain is designed to enforce constant C_2, and if the h is small enough, see also [3].

Now let us show (16). Let $S = S_0 \cup S_\Gamma$ in **A5** (see [5, p. 207]), where $S_\Gamma = \{(x,y)|0 < x < a < 1, 0 < y < 1\}$ and $S_0 = \{(x,y)|a < x < 1, 0 < y < 1\}$ are the singular and the smooth subdomains, respectively. Since the solution in S_0 is smooth enough to have $u \in H^3(S_0)$, we obtain the superconvergence rate $O(h^2)$ from [5]. Moreover, from Theorem 4.1 [5, p. 211] we have all errors from S_0 and S_Γ

Table 4. Errors and condition numbers for $\sigma = \frac{7}{4}$ and the uniform square grids on the stripe domain S^*

N, N_a	32,4	64,8	128,16	256,32	512,64	768,96
$\overline{\|\epsilon\|}_0$	0.324(-5)	0.116(-5)	0.392(-6)	0128(-6)	0.409(-7)	0.208(-7)
$\overline{\|\epsilon\|}_1$	0.118(-3)	0.347(-4)	0.107(-4)	0.355(-5)	0.127(-5)	0.720(-6)
$\max_{ij} \|\epsilon_{ij}\|$	0.184(-4)	0.653(-5)	0.221(-5)	0.730(-6)	0.236(-6)	0.121(-6)
$\max_{ij} \|(\epsilon_x)_{i+\frac{1}{2},j}\|$	0.572(-3)	0.166(-3)	0.577(-4)	0.447(-4)	0.292(-4)	0.221(-4)
$\max_{ij} \|(\epsilon_y)_{i,j+\frac{1}{2}}\|$	0.574(-4)	0.204(-4)	0.693(-5)	0.229(-5)	0.743(-6)	0.381(-6)
Av	0.950(-5)	0.301(-5)	0.955(-6)	0.302(-6)	0.947(-7)	0.479(-7)
Av_x	0.295(-3)	0.847(-4)	0.266(-4)	0.807(-5)	0.245(-7)	0.122(-5)
Av_y	0.286(-4)	0.923(-5)	0.296(-5)	0.940(-6)	0.296(-6)	0.150(-6)
ϵ_1	0.184(-4)	0.592(-5)	0.178(-5)	0.527(-6)	0.156(-6)	0.762(-7)
ϵ_2	0.159(-4)	0.653(-5)	0.212(-5)	0.644(-6)	0.172(-6)	0.943(-7)
$\max_j \|\epsilon_{1j}\|$	0.184(-4)	0.592(-5)	0.178(-5)	0.727(-6)	0.156(-6)	0.762(-7)
$\max_j \|(\epsilon_x)_{\frac{1}{2},j}\|$	0.212(-3)	0.292(-4)	0.577(-4)	0.447(-4)	0.292(-4)	0.221(-4)
$Con(\mathbf{A})$	12.0	50.2	202	812	3249	7311

$$\overline{\|\epsilon\|}_1 = \overline{\|\epsilon\|}_{1,S_0} + \overline{\|\epsilon\|}_{1,S_\Gamma} = O(h^2) + O(h^r), \tag{17}$$

where $r = (p+1)\mu = (p+1)(\sigma - \frac{1}{2})$ and $\sigma \neq 1$. When the uniform square girds are used (i.e., $p = 0$), Eq. (17) leads to

$$\overline{\|\epsilon\|}_1 = O(h^2) + O(h^{\sigma-\frac{1}{2}}), \quad if \quad \frac{1}{2} < \sigma < 2 \wedge \sigma \neq 1. \tag{18}$$

This is (16). In fact, we merge two terms of the right hand side of (16) into one majority term in Theorem 4.1 in [5, p. 211] as $h \to 0$.

Now, let us consider the stripe domain

$$S^* = \{(x,y)\Big| 0 < x < \frac{1}{8}, 0 < y < 1\}, \tag{19}$$

and choose $u(x,y) = x^\sigma(\frac{1}{8} - x)\sin \pi y$ as the solution of the Poisson equation with the homogeneous Dirichlet boundary condition. When the uniform square grids are used, the errors are evaluated for the case of $\sigma = \frac{7}{4}$ only, and listed in Table 4, where N and N_a are the division numbers along axes Y and X, respectively. Then $h = \frac{1}{N} = \frac{1}{8N_a}$. For S^*, the reduced convergence rates (16) remain the same. From Table 4, we see the ratios of $\overline{\|\epsilon\|}_1$ between N and $2N$:

$$\frac{0.118(-3)}{0.347(-4)} = 3.40 = 2^{1.77}, \quad as \quad N = 32. \tag{20}$$

Obviously, the above ratios clearly illustrate a deficit of convergence rates for $\sigma = \frac{7}{4}$.

Suppose that errors $\overline{\|\epsilon\|}_1$ for $\sigma = \frac{7}{4}$ satisfy (16),

$$\overline{\|\epsilon\|}_1 = C_1 h^2 + C_2 h^{\frac{5}{4}}, \tag{21}$$

where $h = \frac{1}{N}$. Table 4 has provided their values at $N = 32, 64, ..., 512$ already. Based on these data, the order $O(h^{\frac{5}{4}})$ in (21) can be confirmed by the Richardson interpolation by excluding the smooth part $O(h^2)$. Moreover, the constants C_1 and C_2 can be obtained from least squares method:

$$C_1 = 0.9034(-1), \quad C_2 = 0.2275(-2), \tag{22}$$

based on the known values of $\overline{\|\epsilon\|}_1$ in Table 4 again. Then Eq. (21) is written as

$$\overline{\|\epsilon\|}_1 = 0.09034 \times h^2 + 0.002275 \times h^{1.25}. \tag{23}$$

The detailed numerical results are omitted.

In summary, the numerical verification on the reduced convergence rate $O(h^{\frac{5}{4}})$ is more difficult than that on $O(h^{\frac{1}{2}})$ in [3], which has been observed numerically when $N = 768$. In this short article, not only is the stripe domain, $S^* = \{(x,y), 0 < x < \frac{1}{8}, 0 < y < 1\}$, chosen, to enlarge the constant C_2 of $h^{\frac{5}{4}}$ in (16), but also the Richardson interpolation is employed to exclude the smooth part $O(h^2)$, and to display clearly $O(h^{\frac{5}{4}})$. This is an accelerating process on convergence of the errors to the singular part of the solution u_h. Without this technique, the verification computation for $O(h^{\frac{5}{4}})$ is exhausted even if a huge computer is available. Moreover, by the least squares method, constants C_1 and C_2 are obtained, and the real errors $\overline{\|\epsilon\|}_1$ of the solution on S^* for $\sigma = \frac{5}{4}$ are governed by (23). Note that the constant in front of $h^{\frac{5}{4}}$ is much smaller than that of h^2, even for the stripe S^*.

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References

[1] Křížek, M., Neittaanmäki, P.: On a global superconvergence of the gradient of linear triangular elements. J. Comput. Appl. Math. *18*, 221–233 (1987).

[2] Křížek, M., Neittaanmäki, P.: Finite element approximation of variational problems and applications. Harlow: Longman 1990.

[3] Li, Z. C.: On non-conforming combination of various finite element methods for solving elliptic boundary value problem. SIAM J. Numer. Anal. *28*, 446–475 (1991).

[4] Li, Z. C.: Combined methods for elliptic equations with singularities, interfaces and infinities, p. 395. Kluwer: Academic Publishers 1998.

[5] Li, Z. C., Hu, H. Y., Fang, Q., Yamamoto, T.: Superconvergence of solution derivatives for the Shortley-Weller difference approximation of the Poisson equation. Part II: Singularity problems. Numer. Funct. Anal. Optim. *24* (*3–4*), 195–221 (2003).

[6] Strang, G., Fix, G. J.: An analysis of the finite element method. Englewood Cliffs, NJ: Prentice-Hall 1973.

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