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ELSEVIER Computers and Mathematics with Applications 51 (2006) 1831–1853
An International Journal
computers & mathematics with applications
www.elsevier.com/locate/camwa

Combinations of Collocation and Finite-Element Methods for Poisson's Equation

HSIN-YUN HU AND ZI-CAI LI*
Department of Applied Mathematics
Department of Computer Science and Engineering
National Sun Yat-sen University
Kaohsiung, Taiwan 80424
National Center for Theoretical Sciences, Taiwan
zcli@math.nsysu.edu.tw

(Received June 2004; revised and accepted October 2005)

Abstract—In this paper, we provide a framework of combinations of collocation method (CM) with the finite-element method (FEM). The key idea is to link the Galerkin method to the least squares method which is then approximated by integration approximation, and led to the CM. The new important uniformly V_h^0-elliptic inequality is proved. Interestingly, the integration approximation plays a role only in satisfying the uniformly V_h^0-elliptic inequality. For the combinations of the finite-element and collocation methods (FEM-CM), the optimal convergence rates can be achieved. The advantage of the CM is to formulate easily linear algebraic equations, where the associated matrices are positive definite but nonsymmetric. We may also solve the algebraic equations of FEM and the collocation equations directly by the least squares method, thus, to greatly improve numerical stability. Numerical experiments are also carried for Poisson's problem to support the analysis. Note that the analysis in this paper is distinct from the existing literature, and it covers a large class of the CM using various admissible functions, such as the radial basis functions, the Sinc functions, etc.

Keywords—Collocation method, Least squares method, Poisson's equation, Finite-element method, Combined method.

1. INTRODUCTION

Since the finite-element method (FEM) today is the most important method among all numerical approaches, owing to wide applications and deep theoretical analysis, we employ the FEM theory in [1,2], to develop the theoretical framework of the collocation methods (CM). If the admissible functions are chosen to be analytical functions, e.g., trigonometric or other orthogonal functions, we may enforce them to satisfy exactly the partial differential equations (PDEs) at certain collocation nodes, by letting the residuals to be zero. This leads to the collocation method. Since the

*Author to whom all correspondence should be addressed.
We thank Professor N. Yan for her valuable suggestions and comments on this paper.

0898-1221/06/$ - see front matter Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX
doi:10.1016/j.camwa.2005.10.018

PDEs and the boundary conditions are copied straightforwardly into the collocation equations, the methods of the paper cover a large class of the collocation method, e.g., those using radial basis functions, the Sinc functions, etc.

The CM is described in a number of books, [3–8]. Here, we also mention several important studies of CM. Bernardi *et al.* [9] provided a coupling finite-element method and spectral method with two kinds of matching conditions on interface. Shen [10–12] gave a series of research study on spectral-Galerkin methods for elliptic equations. Haidvogel [13] applied double Chebyshev polynomials to Poisson's equation. Yin [14] used the Sinc-collocation method to singular Poisson-like problems. Other reports on CM are given by Arnold and Wendland [15], Canuto *et al.* [16], Pathria and Karniadakis [17], and Sneddon [18].

In this paper, we follow the ideas in [6], and provide the combination of the finite-element and collocation methods (FEM-CM). The advantages of this combination are threefold,

(1) flexibility of applications to different geometric shapes and different elliptic equations,
(2) simplicity of computer programming by mimicking the PDEs and the boundary conditions,
(3) varieties of CM using particular solutions, orthogonal polynomials, radial basis functions, the Sinc functions, etc.

Moreover, optimal error bounds are derived, mainly based on the uniformly V_h^0-elliptic inequalities, which are also proved in Sections 4 and 5. Note that the analysis of the CM in this paper is distinct from the existing literature of CM.

This paper is organized as follows. In the next section, the combinations of FEM-CM are described, and in Section 3, linear algebraic equations are formulated, and the solution methods are provided. In Sections 4, the important uniformly V_h^0-elliptic inequality is derived, and in Section 5, the CM involves approximation integrals, and error bounds are derived. In the last section, numerical experiments including a singularity problem are carried out to support the analysis made.

2. COMBINATIONS OF FEMS

Consider Poisson's equation with the Dirichlet condition,

$$-\Delta u = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f(x,y), \qquad \text{in } S, \tag{2.1}$$

$$u|_\Gamma = 0, \qquad \text{on } \Gamma, \tag{2.2}$$

where S is a polygon, and Γ is its boundary. Let S be divided by Γ_0 into two disjoint subregions, S_1 and S_2 (see Figure 1): $S = S_1 \cup S_2 \cup \Gamma_0$ and $S_1 \cap S_2 = \emptyset$. On the interior boundary Γ_0, there exist the interior continuity conditions,

$$u^+ = u^-, \quad u_n^+ = u_n^-, \qquad \text{on } \Gamma_0, \tag{2.3}$$

where $u_n = \frac{\partial u}{\partial n}$, $u^+ = u$ on $\Gamma_0 \cup S_2$ and $u^- = u$ on $\Gamma_0 \cup S_1$. Assume that the solution u in S_2 is smoother than u in S_1. We choose the finite-element method in S_1 and least squares method in S_2, whose discrete forms lead to the CM (see Section 3). Let S_1 be partitioned into small triangles: $\triangle_{ij}$, i.e., $S_1 = \cup_{ij} \triangle_{ij}$. Denote h_{ij} the boundary length of $\triangle_{ij}$. The $\triangle_{ij}$ are said to be

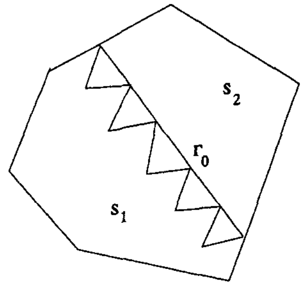

Figure 1. Partition of a polygon.

quasiuniform if $h/\min\{h_{ij}\} \le C$, where $h = \max\{h_{ij}\}$, and C is a constant independent of h. Then, the admissible functions may be expressed by

$$v = \begin{cases} v^- = v_k, & \text{in } S_1, \\ v^+ = \sum\limits_{i=1}^{L} \tilde{a}_i \Psi_i, & \text{in } S_2, \end{cases} \tag{2.4}$$

where $\tilde{a}_i$ are unknown coefficients, and v_k are piecewise k-order Lagrange polynomials in S_1. Assume that $\Psi_i \in C^2(S_2 \cup \partial S_2)$ so that $v^+ \in C^2(S_2 \cup \partial S_2)$. Therefore, we may evaluate (2.1) directly,

$$(\Delta v^+ + f)(Q_i) = 0, \qquad \text{for } Q_i \in S_2, \tag{2.5}$$

at certain collocation nodes $Q_i \in S_2$. Note that v in (2.4) is not continuous on the interior boundary Γ_0. Hence, to satisfy (2.3) we have the interior collocation equations,

$$v^+(Q_i) = v^-(Q_i), \qquad \text{for } P_i \in \Gamma_0, \tag{2.6}$$

$$v_n^+(Q_i) = v_n^-(Q_i), \qquad \text{for } P_i \in \Gamma_0. \tag{2.7}$$

Equations (2.5)–(2.7) are straightforwardly and easy to be formulated. In this paper, we choose the total number of collocation nodes (e.g., P_i) to be larger (or much larger) than the number of unknown coefficients $\tilde{a}_i$. Hence, we may seek the solutions of the entire CM by the least squares method (LSM) in [19], see Remark 2.1 below.

We assume that the solution expansion: $u = \sum_{i=1}^{\infty} a_i \Psi_i$ in S_2 where a_i are the true coefficients. Denote

$$u_L = \sum_{i=1}^{L} a_i \Psi_i, \qquad \text{in } S_2. \tag{2.8}$$

Then, $u = u_L + R_L$, and the remainder,

$$R_L = \sum_{i=L+1}^{\infty} a_i \Psi_i. \tag{2.9}$$

Assume that (2.8) converges exponentially which implies

$$|R_L| = \left| \sum_{i=L+1}^{\infty} a_i \Psi_i \right| = O\left(e^{-\bar{c}L}\right), \qquad \text{in } S_2, \tag{2.10}$$

where $\bar{c} > 0$ and $L > 1$.

Denote by V_h^0 the finite-dimensional collections of (2.4) satisfying $v|_\Gamma = 0$, where we simply assume $\Psi_i|_{\partial S_2 \cap \Gamma} = 0$. If such a condition does not hold, the corresponding collocation equations on $\partial S_2 \cap \Gamma$ are also needed, and the arguments can be provided similarly. The combination of the FEM-CM is designed to seek the approximate solution $u_h \in V_h^0$ such that

$$a(u_h, v) = f(v), \qquad \forall v \in V_h^0, \tag{2.11}$$

where

$$\begin{aligned} a(u,v) = {} & \iint_{S_1} \nabla u \nabla v + \int_{\Gamma_0} u_n^- v^- + P_c \iint_{S_2} \Delta u \Delta v \\ & + \frac{P_c}{h} \int_{\Gamma_0} \left(u^+ - u^-\right)\left(v^+ - v^-\right) + P_c \int_{\Gamma_0} \left(u_n^+ - u_n^-\right)\left(v_n^+ - v_n^-\right), \end{aligned} \tag{2.12}$$

$$f(v) = \iint_{S_1} fv - P_c \iint_{S_2} f \Delta v, \tag{2.13}$$

where $\nabla u = u_x \vec{i} + u_y \vec{j}$, $u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$, $u_n = \frac{\partial u}{\partial n}$, and n is the unit outward normal to ∂S_2. h is the maximal boundary length of Δ_{ij} or $\Box_{ij}$ in S_1, and $P_c > 0$ is chosen to be suitably large but still independent of h.

Denote the the space

$$H^* = \{v, v \in L^2(S),\ v \in H^1(S_1),\ v \in H^1(S_2),\ \Delta v \in L^2(S_2),\ \text{and}\ v|_\Gamma = 0\}, \tag{2.14}$$

accompanied with the norm

$$|||v||| = \left(\|v\|^2_{1,S_1} + P_c \|v\|^2_{1,S_2} + P_c \|\Delta v\|^2_{0,S_2} + \frac{P_c}{h} \|v^+ - v^-\|^2_{0,\Gamma_0} + P_c \|v_n^+ - v_n^-\|^2_{0,\Gamma_0} \right)^{1/2}, \tag{2.15}$$

where $\|v\|_{1,S_1}$ and $\|v\|_{1,S_2}$ are the Sobolev norms. Obviously, $V_h^0 \subset H^*$. For the true solution u to (2.1), we have $a(u - u_h, v) = 0$, $\forall v \in V_h^0$. By means of a traditional argument in [1,20], we have the following theorem easily.

THEOREM 2.1. *Suppose that there exist two inequalities,*

$$a(u,v) \le C|||u||| \times |||v|||, \qquad \forall v \in V_h^0, \tag{2.16}$$

$$a(v,v) \ge C_0 |||v|||^2, \qquad \forall v \in V_h^0, \tag{2.17}$$

where $C_0 > 0$ and C are two constants independent of h and L. Then, the solution of combination (2.11) has the error bound,

$$|||u - u_h||| = C \inf_{v \in V_h^0} |||u - v|||. \tag{2.18}$$

The proof for (2.17) is important but complicated, and is deferred to Section 4. Choose an auxiliary function,

$$u_{I,L} = \begin{cases} u_I, & \text{in } S_1, \\ \sum_{i=1}^{L} a_i \Psi_i, & \text{in } S_2, \end{cases} \tag{2.19}$$

where u_I is the piecewise k-order Lagrange interpolant of the true solution u, and a_i are the true coefficients. Then, $u = \sum_{i=1}^{L} a_i \Psi_i + R_L$ in S_2. By means of the auxiliary function (2.19), we obtain the following corollary easily.

COROLLARY 2.1. *Let all conditions in Theorem 2.1 hold. Suppose that*

$$u \in H^{k+1}(S_1) \qquad \text{and} \qquad u \in H^{k+1}(\Gamma_0). \tag{2.20}$$

Then, there exists the error bound,

$$\begin{aligned} |||u - u_h||| \le C \Big\{ & h^k |u|_{k+1,S_1} + \sqrt{P_c}\, \|R_L\|_{2,S_2} \\ & + \sqrt{P_c} \left(h^{k+1/2} |u|_{k+1,\Gamma_0} + \frac{1}{\sqrt{h}} \|R_L\|_{0,\Gamma_0} + \|(R_L)_n\|_{0,\Gamma_0} \right) \Big\}. \end{aligned} \tag{2.21}$$

Also suppose that the number L of v^+ in (2.4) is chosen such that

$$\begin{aligned} \|R_L\|_{2,S_2} &= O\left(h^k\right), \\ \|R_L\|_{0,\Gamma_0} &= O\left(h^{k+1/2}\right), \\ \|(R_L)_n\|_{0,\Gamma_0} &= O\left(h^k\right). \end{aligned} \tag{2.22}$$

Then, there exists the optimal convergence rate,

$$|||u - u_h||| = O\left(h^k\right). \tag{2.23}$$

REMARK 2.1. The combination (2.11) is nothing new (cf., [6]), except the proof of (2.17) is challenging. Our goal is the CM used in S_2, which can be obtained from (2.11) involving approximation integration. Hence, the combination of Ritz-Galerkin-FEM is a basis for the study in this paper, but more justification will be provided below.

3. LINEAR ALGEBRAIC EQUATIONS OF COMBINATION OF FEM AND CM

Let $\widehat{\iint}_{S_2}$ and $\widehat{\int}_{\Gamma_0}$ denote the approximations $\iint_{S_2}$ and $\int_{\Gamma_0}$ by some integration rules, respectively. The combination of FEM-CM of (2.11) involving integration approximation is given by the following. To seek the approximation solution $\hat{u}_h \in V_h^0$ such that

$$\hat{a}\left(\hat{u}_h, v\right) = \hat{f}\left(v\right), \qquad \forall v \in V_h^0, \tag{3.1}$$

where

$$\begin{aligned} \hat{a}\left(u, v\right) = \iint_{S_1} \nabla u \nabla v + \int_{\Gamma_0} u_n^- v^- + P_c \widehat{\iint}_{S_2} \Delta u \Delta v \\ + \frac{P_c}{h} \widehat{\int}_{\Gamma_0} \left(u^+ - u^-\right)\left(v^+ - v^-\right) + P_c \widehat{\int}_{\Gamma_0} \left(u_n^+ - u_n^-\right)\left(v_n^+ - v_n^-\right), \end{aligned} \tag{3.2}$$

$$\hat{f}(v) = \iint_{S_1} f v - P_c \widehat{\iint}_{S_2} f \Delta v. \tag{3.3}$$

Equation (3.1) can be described equivalently,

$$\hat{a}^*(\hat{u}_h, v) = f_1(v), \qquad \forall v \in V_h^0, \tag{3.4}$$

where

$$\begin{aligned} \hat{a}^*(u, v) = \iint_{S_1} \nabla u \nabla v + \int_{\Gamma_0} u_n^- v^- + P_c \widehat{\iint}_{S_2} (\Delta u + f)(\Delta v + f) \\ + \frac{P_c}{h} \widehat{\int}_{\Gamma_0} (u^+ - u^-)(v^+ - v^-) + P_c \widehat{\int}_{\Gamma_0} (u_n^+ - u_n^-)(v_n^+ - v_n^-), \end{aligned} \tag{3.5}$$

$$f_1(v) = \iint_{S_1} f v. \tag{3.6}$$

In S_2, we choose the integration rules,

$$\widehat{\iint}_{S_2} g^2 = \sum_{ij} \alpha_{ij} g^2(Q_{ij}), \qquad Q_{ij} \in S_2, \tag{3.7}$$

$$\widehat{\int}_{\Gamma_0} g^2 = \sum_{j} \alpha_j g^2(Q_j), \qquad Q_j \in \Gamma_0, \tag{3.8}$$

where α_{ij} and α_j are positive weights. In fact, we may formulate the collocation equations at $Q_{ij} \in S_2$, and $Q_j \in \Gamma_0$ directly. The collocation equations at Q_{ij}, and Q_j are given by

$$\left(\Delta v^+ + f\right)\left(Q_{ij}\right) = 0, \qquad Q_{ij} \in S_2, \tag{3.9}$$

$$\left(v^+ - v^-\right)\left(Q_j\right) = 0, \qquad Q_j \in \Gamma_0, \tag{3.10}$$

$$\left(v_n^+ - v_n^-\right)\left(Q_j\right) = 0, \qquad Q_j \in \Gamma_0. \tag{3.11}$$

By introducing suitable weight functions, we rewrite the equations (3.9)–(3.11) as

$$\sqrt{P_c \alpha_{ij}}\left(\Delta v^+ + f\right)\left(Q_{ij}\right) = 0, \qquad Q_{ij} \in S_2, \tag{3.12}$$

$$\sqrt{\frac{P_c \alpha_j}{2h}}\left(v^+ - v^-\right)\left(Q_j\right) = 0, \qquad Q_j \in \Gamma_0, \tag{3.13}$$

$$\sqrt{\frac{P_c \alpha_j}{2}}\left(v_n^+ - v_n^-\right)\left(Q_j\right) = 0, \qquad Q_j \in \Gamma_0, \tag{3.14}$$

where α_{ij} are positive weights, and Q_j are the interior element nodes of Γ_0. We give some rules of integration with explicit weights α_j and α_{ij} in (3.12). First, choose the trapezoidal rule,

$$\widehat{\int}_{\overline{Q_1Q_2}} g^2 = \frac{\overline{Q_1Q_2}}{2}\left(g^2(Q_1)+g^2(Q_2)\right) = \frac{H}{2}\left(g^2(Q_1)+g^2(Q_2)\right). \tag{3.15}$$

The weights $\alpha_j = H/2$ or H when Q_j is at a corner of ∂S or not. Let S_2 be a rectangular subdomain of S, and be divided into uniformly difference grids with the meshspacing H, where Q_{ij} denote the collocation nodes (i,j). Hence, the weights α_{ij} in (3.12) have the following values,

$$\alpha_{ij} = \begin{cases} H^2, \ (i,j) \in S_2, \\ \frac{1}{2}H^2, & (i,j) \in \partial S_2 \text{ excluding corners of } \partial S_2, \\ \frac{1}{4}H^2, & (i,j) \in \text{ corners of } \partial S_2. \end{cases} \tag{3.16}$$

We may choose more efficient rules, such as the Legendre-Gauss rule with two boundary nodes fixed in [7,21],

$$\widehat{\int_{-1}^{1}} g^2(x)\,dx = \sum_{j=1}^{n} w_j g^2(x_j), \tag{3.17}$$

where x_j is the jth zero of $P_n(x)$, and $P_n(x)$ are the Legendre polynomials defined by

$$P_n(x) = \frac{(-1)^n}{2^n n!}\cdot\frac{d^n}{dx^n}\left[\left(1-x^2\right)^n\right], \qquad n \geq 1. \tag{3.18}$$

The weights are given by

$$w_j = \frac{2}{(1-x_j)\left[P_n'(x_j)\right]^2}. \tag{3.19}$$

Then when choosing the collocation nodes $Q_{ij} = (x_i, y_j)$, the weights in (3.12) are obtained as

$$\alpha_{ij} = w_i w_j, \qquad (i,j) \in S_2. \tag{3.20}$$

Let $f = g^2$ and $f \in C^{2n}[-1,1]$, then the remainder of (3.17) is given by

$$E(f) = \frac{2^{2n+1}[n!]^4}{(2n+1)[(2n)!]^3} f^{2n}(\xi), \qquad f = g^2, \quad -1 < \xi < 1. \tag{3.21}$$

Let g in $[-1,1]$ be polynomials of order L. Then, $f(= g^2)$ is polynomials of order $2L$. Choose $n = L+1$, then the derivatives $f^{2n}(\xi) = f^{(2L+2)} \equiv 0$ and $E(f) \equiv 0$. Therefore, when S_2 is a rectangle, the functions v^+ in S_2 are chosen to be polynomials of order L. Functions Δv^+ are polynomials of order $L-2$. The Legendre-Gauss rule with $n = L-1$ in (3.7) and (3.20) offers no errors for $\widehat{\iint}_{S_2}(\Delta v^+)^2$, i.e.,

$$\widehat{\iint}_{S_2} (\Delta v^+)^2 = \iint_{S_2} (\Delta v^+)^2. \tag{3.22}$$

Now, let us establish the linear algebraic equations of combination (3.4) of FEM-CM. First, consider the entire FEM in S_1 only,

$$a_1(\hat{u}_h, v) = f_1(v), \qquad \forall v \in V_h, \tag{3.23}$$

where

$$a_1(u,v) = \iint_{S_1} \nabla u \nabla v + \int_{\Gamma_0} u_n^- v^-, \qquad f_1(v) = \iint_{S_1} fv. \tag{3.24}$$

We obtain the linear algebraic equations,

$$\mathbf{A}_1 \vec{x}_1 = \vec{b}_1, \tag{3.25}$$

where $\vec{x}_1$ is a vector consisting of v_{ij} only, and matrix $\mathbf{A}_1$ is nonsymmetric.

Next, equations (3.12)–(3.14) in $S_2 \cup \Gamma_0$ are denoted by

$$\mathbf{A}_2 \vec{x}_2 = \vec{b}_2, \tag{3.26}$$

where $\vec{x}_2$ is a vector consisting of $\tilde{a}_i$, v_{1j} and v_{0j}, and v_{0j} and v_{1j} are the unknowns on the two boundary layer nodes in S_1 close to Γ_0 if the linear FEM is used. Denote by M_1 the number of all collocation nodes in S_2 and ∂S_2, and by N_1 the number of v_{1j} and v_{0j}. Matrix $\mathbf{A}_2 \in R^{M_1 \times (L+N_1)}$. Therefore, we can see

$$\begin{aligned} &\frac{1}{2} \vec{x}_2^\top \mathbf{A}_2^\top \mathbf{A}_2 \vec{x}_2 - \mathbf{A}_2^\top \vec{b}_2 \vec{x}_2 + \vec{c} \\ &= \frac{P_c}{2} \widehat{\iint_{S_2}} (\Delta v + f)^2 + \frac{P_c}{2h} \widehat{\int_{\Gamma_0}} \left(v^+ - v^-\right)^2 + \frac{P_c}{2} \widehat{\int_{\Gamma_0}} \left(v_n^+ - v_n^-\right)^2. \end{aligned} \tag{3.27}$$

Combining (3.25) and (3.27) yields explicitly[1]

$$\mathbf{A} \vec{x} = \vec{b}, \tag{3.28}$$

$$\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2^\top \mathbf{A}_2, \qquad \vec{b} = \vec{b}_1 + \mathbf{A}_2^\top \vec{b}_2, \tag{3.29}$$

where $\vec{x}$ is a vector consisting of the coefficients $\tilde{a}_i$ and v_{ij} in $S_1 \cup \Gamma_0$. Denote by N the number of nodes on $S_1 \cup \Gamma_0$, then the vector $\vec{x}$ in (3.28) has $N + L$ dimensions. The matrix $\mathbf{A}$ is nonsymmetric, but positive definite, based on Theorem 5.1 given later.

Let us briefly address the solution methods for (3.28). When P_c is chosen large enough, matrix $\mathbf{A} \in R^{(L+N)\times(L+N)}$ in (3.28) is positive definite, nonsymmetric and sparse when $N \gg L$. When $L + N$ is not huge, we may choose the Gaussian elimination without pivoting to solve (3.28), see [19].

Also, since $\hat{a}(u - \hat{u}_h, v) = 0$, $\forall v \in V_h^0$, we obtain the following theorem.

THEOREM 3.1. *Suppose that there exist two inequalities,*

$$\hat{a}(u,v) \le C|||u||| \times |||v|||, \qquad \forall v \in V_h^0, \tag{3.30}$$

$$\hat{a}(v,v) \ge C_0|||v|||^2, \qquad \forall v \in V_h^0, \tag{3.31}$$

where $C_0 > 0$ and C are two constants independent of h and L. Then, the solution of combination (3.1) has the error bound,

$$|||u - \hat{u}_h||| \le C \inf_{v \in V_h} |||u - v|||. \tag{3.32}$$

Moreover, the optimal convergence rate (2.23) holds if the conditions (2.20) and (2.22) are satisfied.

[1] In (3.29), (3.34), and (3.40), the dimensions of matrices may not be consistent, where the equality means that the matrices of less dimensions should expand by filling up more zero entices.

The proof of inequality (3.31) is deferred to Section 5.

REMARK 3.1. Note that equation (3.28), called Method I, presents exactly combination (3.4). There arises a question. Since (3.28) results from (3.25) and (3.26), should we solve (3.25) and (3.26) directly (i.e., together) by the least squares method? The following arguments give a positive justification.

METHOD I. We rewrite (3.28) and (3.29) as

$$\mathbf{A}\vec{y} = \vec{b}, \tag{3.33}$$

i.e.,

$$\mathbf{A}_1\vec{y} + \mathbf{A}_2^\top \mathbf{A}_2\vec{y} = \vec{b_1} + \mathbf{A}_2^\top \vec{b_2}. \tag{3.34}$$

METHOD II. THE LEAST SQUARES METHOD DIRECTLY. Solve

$$\mathbf{A}_1\vec{x} = \vec{b_1}, \ \mathbf{A}_2\vec{x} = \vec{b_2}, \tag{3.35}$$

by

$$I(\vec{x}) = \min_{\vec{z}} I(\vec{z}), \tag{3.36}$$

where

$$I(\vec{z}) = \|\mathbf{A}_1\vec{z} - \vec{b_1}\|^2 + \|\mathbf{A}_2\vec{z} - \vec{b_2}\|^2, \tag{3.37}$$

and $\|\cdot\|$ is the Euclidean norm.

PROPOSITION 3.1. *Let $\vec{x}$ and $\vec{y}$ be the solutions from (3.35) and (3.33) respectively, then $\vec{x} \approx \vec{y}$ with the relative error bound,*

$$\frac{\|\vec{x} - \vec{y}\|}{\|\vec{y}\|} \le \text{Cond.}\,(\mathbf{A}) \frac{\|\vec{r}\|}{\left\|\vec{b}\right\|} \le \text{Cond.}\,(\mathbf{A})\,(1 + \|\mathbf{A}_2\|)\,\frac{\varepsilon}{\left\|\vec{b}\right\|}, \tag{3.38}$$

where the error of Method II is

$$\varepsilon = \left(\left\|\mathbf{A}_1\vec{x} - \vec{b_1}\right\|^2 + \left\|\mathbf{A}_2\vec{x} - \vec{b_2}\right\|^2\right)^{1/2},$$

Cond. $(\mathbf{A})$ *denotes the condition number of* $\mathbf{A}$*:*

$$\text{Cond.}\ (\mathbf{A}) = \sqrt{\frac{\lambda_{\max}(\mathbf{A}^\top\mathbf{A})}{\lambda_{\min}(\mathbf{A}^\top\mathbf{A})}},$$

and $\lambda_{\max}(\mathbf{A}\top\mathbf{A})$ and $\lambda_{\max}(\mathbf{A}\top\mathbf{A})$ are the maximal and minimal eigenvalues of $\mathbf{A}\top\mathbf{A}$, respectively.

PROOF. Denote $I(\vec{x}) = \varepsilon^2$, we then obtain (3.36)

$$\left\|\mathbf{A}_1\vec{x} - \vec{b_1}\right\| \le \varepsilon, \qquad \left\|\mathbf{A}_2\vec{x} - \vec{b_2}\right\| \le \varepsilon. \tag{3.39}$$

Consider the remainder of (3.34) when $\vec{x}$ replaces $\vec{y}$:

$$\vec{r} = \mathbf{A}\vec{x} - \vec{b} = \mathbf{A}_1\vec{x} - \vec{b_1} + \mathbf{A}\top_2\mathbf{A}_2\vec{x} - \mathbf{A}\top_2\vec{b_2}. \tag{3.40}$$

We have from (3.39)

$$\|\vec{r}\| \le \left\|\mathbf{A}_1\vec{x} - \vec{b_1}\right\| + \|\mathbf{A}_2\| \left\|\mathbf{A}_2\vec{x} - \vec{b_2}\right\| \le (1 + \|\mathbf{A}_2\|)\,\varepsilon. \tag{3.41}$$

Moreover, we have from equations (3.33) and (3.40)

$$\mathbf{A}\,(\vec{x} - \vec{y}) = \vec{r}.$$

Since $\|\vec{y}\| \geq \|\vec{b}\|/\|\mathbf{A}\|$ from (3.33), the desired result (3.38) is obtained by following [19,21]. Since ε is very small, the solution $\vec{x}$ of Method II is the solution $\vec{y}$ of Method I approximately. This completes the proof of Proposition 3.1. ∎

4. UNIFORM V_h^0-ELLIPTIC INEQUALITY

The key analysis of combinations (2.11) and (3.1) is to prove the uniform V_h^0-elliptic inequalities (2.17) and (3.31), since the proof for (2.16) and (3.30) is much simpler. We shall prove (2.17) in this section and then (3.31) in the next section.

First, we consider $a(v,v)$ without the term $\int_{\Gamma_0} v_n^- v^-$. Define the norms

$$\|v\|_E = \left(|v|^2_{1,S_1} + P_c \|\Delta v\|^2_{0,S_2} + \frac{P_c}{h} \left\|v^+ - v^-\right\|^2_{0,\Gamma_0} + P_c \left\|v_n^+ - v_n^-\right\|^2_{0,\Gamma_0} \right)^{1/2}, \tag{4.1}$$

and

$$\overline{\|v^+ - v^-\|}_{\ell,\Gamma_0} = \left\|\hat{v}^+ - v^-\right\|_{\ell,\Gamma_0}, \tag{4.2}$$

where $\ell = 0, 1/2$, and $\hat{v}^+$ is the piecewise k-order polynomial interpolant of v^+ in S_2. Then, we have the following lemma.

LEMMA 4.1. *Suppose that there exists a positive constant $\nu(>0)$ such that*

$$\left\|v^+\right\|_{\ell,\Gamma_0} \leq CL^{\ell\nu} \left\|v^+\right\|_{0,\Gamma_0}, \qquad \ell = 1,2,\dots \tag{4.3}$$

Then, there exists the bound for $v \in V_h^0$,

$$\left\|v^+ - v^-\right\|_{1/2,\Gamma_0} \leq \frac{C}{\sqrt{h}} \left\|v^+ - v^-\right\|_{0,\Gamma_0} + Ch^{3/2}L^{2\nu} \left\|v^+\right\|_{1,S_2}. \tag{4.4}$$

PROOF. We have from triangle inequalities,

$$\begin{aligned} \left\|v^+ - v^-\right\|_{1/2,\Gamma_0} &\leq \overline{\|v^+ - v^-\|}_{1/2,\Gamma_0} + \left\|\hat{v}^+ - v^+\right\|_{1/2,\Gamma_0}, \\ \overline{\|v^+ - v^-\|}_{0,\Gamma_0} &\leq \left\|v^+ - v^-\right\|_{0,\Gamma_0} + \left\|\hat{v}^+ - v^+\right\|_{0,\Gamma_0}. \end{aligned} \tag{4.5}$$

Then from the inverse inequality for piecewise polynomials, there exists the bound,

$$\begin{aligned} \left\|v^+ - v^-\right\|_{1/2,\Gamma_0} &\leq \overline{\|v^+ - v^-\|}_{1/2,\Gamma_0} + \left\|\hat{v}^+ - v^+\right\|_{1/2,\Gamma_0} \\ &\leq \frac{C}{\sqrt{h}} \overline{\|v^+ - v^-\|}_{0,\Gamma_0} + \left\|\hat{v}^+ - v^+\right\|_{1/2,\Gamma_0} \\ &\leq \frac{C}{\sqrt{h}} \left\|v^+ - v^-\right\|_{0,\Gamma_0} + \frac{C}{\sqrt{h}} \left\|\hat{v}^+ - v^+\right\|_{0,\Gamma_0} + \left\|\hat{v}^+ - v^+\right\|_{1/2,\Gamma_0}. \end{aligned} \tag{4.6}$$

Moreover, from (4.3) we have

$$\begin{aligned} h^{-1/2} \left\|\hat{v}^+ - v^+\right\|_{0,\Gamma_0} + \left\|\hat{v}^+ - v^+\right\|_{1/2,\Gamma_0} &\leq Ch^{3/2} \left\|v^+\right\|_{2,\Gamma_0} \\ &\leq Ch^{3/2}L^{2\nu} \left\|v^+\right\|_{0,\Gamma_0} \leq Ch^{3/2}L^{2\nu} \left\|v^+\right\|_{1,S_2}. \end{aligned} \tag{4.7}$$

Combining (4.6) and (4.7) yields the desired result (4.4). This completes the proof of Lemma 4.1. ∎

LEMMA 4.2. *There exist the bounds for* $v \in V_h^0$,

$$\|v^+\|_{1,S_2} \le C\{\|\Delta v^+\|_{-1,S_2} + \|v^+\|_{1/2,\partial S_2}\}, \tag{4.8}$$

$$\|v_n^+\|_{-1/2,\Gamma_0} \le C\{\|\Delta v^+\|_{-1,S_2} + \|v^+\|_{1/2,\partial S_2}\}, \tag{4.9}$$

where C is a constant independent of h and L.

PROOF. We cite the bound from [2, pp. 189–192],

$$\|u\|^2_{\tilde{H}^{s,2m}(\Omega)} \le C\{\|Au\|^2_{H^{s-2m}(\Omega)} + \sum_{k=0}^{m-1} \|B_k u\|^2_{H^{s-g_k-1/2}(\partial\Omega)}\}, \tag{4.10}$$

where $s < 2m$, and m is a positive integer. The notations are: $Au = \Delta u$, $B_0 u = u$, $B_1 u = u_n$, $g_0 = 0$ and $g_1 = 1$. The norm on the left-hand side in (4.10) is defined in [2, p. 183],

$$\|u\|^2_{\tilde{H}^{s,r}(\Omega)} = \|u\|^2_{H^s(\Omega)} + \sum_{k=0}^{r-1} \|D_n^k u\|^2_{H^{s-k-1/2}(\partial\Omega)}, \tag{4.11}$$

where r is a positive integer, s is an integer, and $D_n^k = \frac{\partial^k}{\partial n^k}$ is the kth normal derivatives. In (4.10), choosing $s = 1$, $m = 1$ and $\Omega = S_2$, we obtain

$$\|u\|^2_{\tilde{H}^{1,2}(S_2)} \le C\{\|\Delta u\|^2_{H^{-1}(S_2)} + \|u\|^2_{H^{1/2}(\partial S_2)}\}, \tag{4.12}$$

where the norm is given in (4.12) with $s = 1$, $r = 2$, and $\Omega = S_2$,

$$\|u\|^2_{\tilde{H}^{1,2}(S_2)} = \|u\|^2_{H^1(S_2)} + \|u\|^2_{H^{1/2}(\partial S_2)} + \|u_n\|^2_{H^{-1/2}(\partial S_2)}. \tag{4.13}$$

Combining (4.10) and (4.13) gives the following bound,

$$\|u\|^2_{1,S_2} + \|u\|^2_{1/2,\partial S_2} + \|u_n\|^2_{-1/2,\partial S_2} \le C\{\|\Delta u\|^2_{-1,S_2} + \|u\|^2_{1/2,\partial S_2}\}. \tag{4.14}$$

The desired results (4.8) and (4.9) are obtained directly from (4.14). This completes the proof of Lemma 4.2. ∎

LEMMA 4.3. *Let (4.3) and the following bound hold,*

$$h^{3/2} L^{2\nu} = o(1). \tag{4.15}$$

Then, for $v \in V_h^0$ *there exists the bound,*

$$\|v^+\|_{1,S_2} \le C\left\{\|v^-\|_{1/2,\Gamma_0} + \frac{1}{\sqrt{h}}\|v^+ - v^-\|_{0,\Gamma_0} + \|\Delta v^+\|_{0,S_2}\right\}, \tag{4.16}$$

where C is a constant independent of h and L.

PROOF. From Lemma 4.1, we have

$$\begin{aligned}\|v^+\|_{1/2,\Gamma_0} &\le \|v^-\|_{1/2,\Gamma_0} + \|v^+ - v^-\|_{1/2,\Gamma_0} \\ &\le \|v^-\|_{1/2,\Gamma_0} + \frac{C}{\sqrt{h}}\|v^+ - v^-\|_{0,\Gamma_0} + C h^{3/2} L^{2\nu} \|v^+\|_{1,S_2}.\end{aligned} \tag{4.17}$$

From Lemma 4.2 and (4.17)

$$\begin{aligned}\|v^+\|_{1,S_2} &\le C\left\{\|v^+\|_{1/2,\Gamma_0} + \|\Delta v^+\|_{-1,S_2}\right\} \\ &\le C\left\{\|v^-\|_{1/2,\Gamma_0} + \frac{1}{\sqrt{h}}\|v^+ - v^-\|_{0,\Gamma_0} + h^{3/2} L^{2\nu}\|v^+\|_{1,S_2} + \|\Delta v^+\|_{0,S_2}\right\}.\end{aligned} \tag{4.18}$$

This leads to

$$\left\|v^+\right\|_{1,S_2} \le \frac{C}{1 - Ch^{3/2}L^{2\nu}} \left\{ \left\|v^-\right\|_{1/2,\Gamma_0} + \frac{1}{\sqrt{h}} \left\|v^+ - v^-\right\|_{0,\Gamma_0} + \left\|\Delta v^+\right\|_{0,S_2} \right\}. \tag{4.19}$$

The desired result (4.16) follows from $Ch^{3/2}L^{2\nu} \le 1/2$ by assumption (4.15). This completes the proof of Lemma 4.3. ∎

LEMMA 4.4. *Let $\Gamma \cap S_1 \neq \emptyset$, (4.3) and (4.15) hold, there exists an inequality*

$$C_0|||v||| \le \|v\|_E, \qquad \forall\, v \in V_h^0, \tag{4.20}$$

where $|||v|||$ and $\|v\|_E$ are defined in (2.15) and (4.1) respectively, and $C_0 > 0$ has a lower bound independent of h and L.

PROOF. By the contradiction, we can find a sequence $\{v_\ell\} \subseteq H^*$ such that

$$|||v_\ell||| = 1, \quad \|v_\ell\|_E \to 0, \qquad \text{as } \ell \to \infty. \tag{4.21}$$

First, $\|v_\ell\|_E \to 0$ implies that for large ℓ, $|v_\ell^-|_{1,S_1} \le 1$ and $v_\ell^-\big|_{S_1\cap\Gamma} = 0$, and then $\|v_\ell^-\|_{1,S_1}$ is bounded. Based on the Kandrosov or Rellich theorem [1], there exists a subsequence $\{v_\ell^-\}$ in $L^2(S_1)$ (also written as $\{v_\ell^-\}$) such that $v_\ell^- \to \bar{v}^- \in L^2(S_1)$. Then, $\bar{v}^- \in H^1(S_1)$, since $|\bar{v}^-|_{1,S_1}$ are bounded due to $|v_\ell^-|_{1,S_1} \le 1$. Moreover, $\|v_\ell\|_E \to 0$ gives $|v_\ell^-|_{1,S_1} \to 0$ as $\ell \to \infty$. Since $H^1(S_1)$ is complete, we conclude that $|\bar{v}^-|_{1,S_1} = \lim_{\ell\to\infty} |v_\ell^-|_{1,S_1} = 0$. Hence $\bar{v}^-$ is a constant, and $\bar{v}^- \equiv 0$ in S_1 due to $\bar{v}^-\big|_{S_1\cap\Gamma} = 0$.

From the trace theorem [1],

$$\left\|v_\ell^-\right\|_{1/2,\Gamma_0} \le C \left\|v_\ell^-\right\|_{1,S_1}, \tag{4.22}$$

$\|v_\ell^-\|_{1/2,\Gamma_0}$ is also bounded, and

$$\lim_{\ell\to\infty} \left\|v_\ell^-\right\|_{1/2,\Gamma_0} = \left\|\bar{v}^-\right\|_{1/2,\Gamma_0} = 0. \tag{4.23}$$

Next, consider the sequence v_ℓ^+ in S_2. We have from Lemma 4.3,

$$\begin{aligned} \left\|v_\ell^+\right\|_{1/2,\Gamma_0} &\le C \left\|v_\ell^+\right\|_{1,S_2} \\ &\le C \left\{ \left\|v_\ell^-\right\|_{1/2,\Gamma_0} + \frac{1}{\sqrt{h}} \left\|v_\ell^+ - v_\ell^-\right\|_{0,\Gamma_0} + \left\|\Delta v_\ell^+\right\|_{0,S_2} \right\}. \end{aligned} \tag{4.24}$$

We conclude that $\|v_\ell^+\|_{1/2,\Gamma_0}$ is bounded, and that $\lim_{\ell\to\infty} \|v_\ell^+\|_{1/2,\Gamma_0} = 0$ from (4.24), (4.23) and $\|v_\ell\|_E \to 0$. Based on Lemma 4.2,

$$\begin{aligned} \left\|v_\ell^+\right\|_{1,S_2} &\le C \left\{ \left\|\Delta v_\ell^+\right\|_{-1,S_2} + \left\|v_\ell^+\right\|_{1/2,\partial S_2} \right\} \\ &\le C \left\{ \left\|\Delta v_\ell^+\right\|_{0,S_2} + \left\|v_\ell^+\right\|_{1/2,\Gamma_0} \right\}. \end{aligned} \tag{4.25}$$

Hence $\|v_\ell^+\|_{1,S_2}$ is also bounded from $\|v_\ell\|_E \to 0$, and then $\lim_{\ell\to\infty}\|v_\ell^+\|_{1,S_2} = 0$. By repeating the above arguments, there exists also a subsequence v_ℓ^+ to converge $\bar v^+ \in H^1(S_2)$. Moreover, we have $\|\bar v^+\|_{1,S_2} = \lim_{\ell\to\infty}\|v_\ell^+\|_{1,S_2} = 0$, and then $\bar v^+ \equiv 0$ in S_2. Hence, $\bar v \equiv 0$ in the entire S and $|||\bar v||| = 0$. This contradicts the assumption $|||\bar v||| = \lim_{\ell\to\infty}|||v_\ell||| = 1$ in (4.21), and completes the proof of Lemma 4.4. ∎

Now, we give the main theorem.

THEOREM 4.1. *Let $\Gamma \cap \partial S_1 \neq \emptyset$, (4.3) and (4.15) hold, and P_c be chosen to be suitably large but still independent of h. Then, the uniformly V_h^0- elliptic inequality holds,*

$$C_0|||v|||^2 \le a(v,v), \qquad \forall\, v \in V_h^0, \tag{4.26}$$

where $C_0 > 0$ is a constant independent of h and L.

PROOF. From Lemma 4.4, we obtain the bound,

$$\begin{aligned} a(v,v) &\ge \|v\|_E^2 - \int_{\Gamma_0} v_n^- v^- \ge C_1|||v|||^2 - \int_{\Gamma_0} v_n^- v^- \\ &= C_1\left(\|v\|_{1,S_1}^2 + P_c\|v\|_{1,S_2}^2 + P_c\|\Delta v\|_{0,S_2}^2 + \frac{P_c}{h}\left\|v^+ - v^-\right\|_{0,\Gamma_0}^2 + P_c\left\|v_n^+ - v_n^-\right\|_{0,\Gamma_0}^2\right) \\ &\quad - \int_{\Gamma_0} v_n^- v^-, \end{aligned} \tag{4.27}$$

where $C_1 > 0$ has a lower bound independent of h and L. Next, we have

$$\left|\int_{\Gamma_0} v_n^- v^-\right| \le \|v_n^-\|_{-1/2,\Gamma_0}\|v^-\|_{1/2,\Gamma_0}. \tag{4.28}$$

Moreover, there exist the bounds for $v \in V_h^0$,

$$\left\|v^-\right\|_{1/2,\Gamma_0} \le C\left\|v^-\right\|_{1,S_1}, \tag{4.29}$$

$$\begin{aligned} \left\|v_n^-\right\|_{-1/2,\Gamma_0} &\le \left\|v_n^+\right\|_{-1/2,\Gamma_0} + \left\|v_n^+ - v_n^-\right\|_{-1/2,\Gamma_0} \\ &\le C\left\{\left\|v^+\right\|_{1,S_2} + \left\|\Delta v^+\right\|_{0,S_2} + \left\|v_n^+ - v_n^-\right\|_{0,\Gamma_0}\right\}, \end{aligned} \tag{4.30}$$

where we have used the bound from Lemma 4.2,

$$\begin{aligned} \left\|v_n^+\right\|_{-1/2,\Gamma_0} &\le C\left\{\left\|\Delta v^+\right\|_{-1,S_2} + \left\|v^+\right\|_{1/2,\partial S_2}\right\} \\ &\le C\left\{\left\|\Delta v^+\right\|_{0,S_2} + \left\|v^+\right\|_{1,S_2}\right\}. \end{aligned}$$

Since $Cab \le \epsilon a^2 + (C^2/4\epsilon)b^2$ for any $\epsilon > 0$, we obtain from (4.29) and (4.30)

$$\begin{aligned} \left|\int_{\Gamma_0} v_n^- v^-\right| &\le C\left\|v^-\right\|_{1,S_1}\left\{\left\|v^+\right\|_{1,S_2} + \left\|\Delta v^+\right\|_{0,S_2} + \left\|v_n^+ - v_n^-\right\|_{0,\Gamma_0}\right\} \\ &\le \frac{C_1}{2}\left\|v^-\right\|_{1,S_1}^2 + \frac{C^2}{2C_1}\left\{\left\|v^+\right\|_{1,S_2} + \left\|\Delta v^+\right\|_{0,S_2} + \left\|v_n^+ - v_n^-\right\|_{0,\Gamma_0}\right\}^2 \\ &\le \frac{C_1}{2}\left\|v^-\right\|_{1,S_1}^2 + \frac{3C^2}{2C_1}\left\{\left\|v^+\right\|_{1,S_2}^2 + \left\|\Delta v^+\right\|_{0,S_2}^2 + \left\|v_n^+ - v_n^-\right\|_{0,\Gamma_0}^2\right\}, \end{aligned} \tag{4.31}$$

where C_1 is given in (4.27). Combining (4.27) and (4.31) gives

$$\begin{aligned} a(v,v) &\ge \frac{C_1}{2}\|v\|_{1,S_1}^2 \\ &\quad + \left(C_1P_c - \frac{3C^2}{2C_1}\right)\left(\|v\|_{1,S_2}^2 + \|\Delta v\|_{0,S_2}^2 + \left\|v_n^+ - v_n^-\right\|_{0,\Gamma_0}^2\right) + C_1\frac{P_c}{h}\left\|v^+ - v^-\right\|_{0,\Gamma_0}^2 \\ &\ge \frac{C_1}{2}|||v|||^2, \end{aligned} \tag{4.32}$$

provided that $C_1P_c - (3C^2/2C_1) \ge (1/2)C_1P_c$. This leads to $P_c \ge 3(C^2/C_1^2)$, which is suitably large but still independent of h and L. Then the uniformly V_h^0-elliptic inequality (4.26) holds with $C_0 = C_1/2$. This completes the proof of Theorem 4.1. ∎

5. UNIFORM V_h^0-ELLIPTIC INEQUALITY INVOLVING INTEGRATION APPROXIMATION

In this section, we prove the uniformly V_h^0-elliptic inequality, (3.31). Choose the integration rule,

$$\widehat{\int}_{\Gamma_0} v^2 = \int_{\Gamma_0} \hat{v}^2 = \overline{\|v\|}^2_{0,\Gamma_0}, \tag{5.1}$$

where $\hat{v}$ is the k-order interpolant of v. First, we give a few lemmas.

LEMMA 5.1. *Let (4.3) and*

$$\left\|v_n^+\right\|_{1,\Gamma_0} \le CL^{2\nu} \left\|v^+\right\|_{1,S_2}, \qquad \forall\, v \in V_h^0 \tag{5.2}$$

hold, where $\nu(>0)$ is a positive constant. There exist the bounds for $v \in V_h^0$,

$$\overline{\left\|v^+ - v^-\right\|}_{0,\Gamma_0} \ge \left\|v^+ - v^-\right\|_{0,\Gamma_0} - Ch^2L^{2\nu} \left\|v^+\right\|_{1,S_2}, \tag{5.3}$$

$$\overline{\left\|v_n^+ - v_n^-\right\|}_{0,\Gamma_0} \ge \left\|v_n^+ - v_n^-\right\|_{0,\Gamma_0} - ChL^{2\nu} \left\|v^+\right\|_{1,S_2}, \tag{5.4}$$

where C is a constant independent of h and L.

PROOF. We have

$$\left\|v^+ - v^-\right\|_{0,\Gamma_0} \le \overline{\left\|v^+ - v^-\right\|}_{0,\Gamma_0} + \left\|\hat{v}^+ - v^+\right\|_{0,\Gamma_0}, \tag{5.5}$$

and from (4.3)

$$\left\|\hat{v}^+ - v^+\right\|_{0,\Gamma_0} \le Ch^2 \left|v^+\right|_{2,\Gamma_0} \le Ch^2L^{2\nu} \|v\|_{0,\Gamma_0} \le Ch^2L^{2\nu} \|v\|_{1,S_2}. \tag{5.6}$$

Then combining (5.5) and (5.6) gives the first desired bound (5.3),

$$\overline{\left\|v^+ - v^-\right\|}_{0,\Gamma_0} \ge \left\|v^+ - v^-\right\|_{0,\Gamma_0} - Ch^2L^{2\nu} \|v\|_{1,S_2}. \tag{5.7}$$

Similarly, we obtain from (5.2)

$$\begin{aligned}\overline{\left\|v_n^+ - v_n^-\right\|}_{0,\Gamma_0} &\ge \left\|v_n^+ - v_n^-\right\|_{0,\Gamma_0} - \left\|v_n^+ - \hat{v}_n^+\right\|_{0,\Gamma_0} \\ &\ge \left\|v_n^+ - v_n^-\right\|_{0,\Gamma_0} - Ch \|v_n\|_{1,\Gamma_0} \\ &\ge \left\|v_n^+ - v_n^-\right\|_{0,\Gamma_0} - ChL^{2\nu} \|v\|_{1,S_2}.\end{aligned} \tag{5.8}$$

This is the second desired bound (5.4), and completes Lemma 5.1. ∎

LEMMA 5.2. *Let all conditions in Lemma 5.1 hold. Then,*

$$\overline{\left\|v^+ - v^-\right\|}^2_{0,\Gamma_0} \ge \frac{1}{2}\left\|v^+ - v^-\right\|^2_{0,\Gamma_0} - Ch^4L^{4\nu} \left\|v^+\right\|^2_{1,S_2}, \tag{5.9}$$

$$\overline{\left\|v_n^+ - v_n^-\right\|}^2_{0,\Gamma_0} \ge \frac{1}{2}\left\|v_n^+ - v_n^-\right\|^2_{0,\Gamma_0} - Ch^2L^{4\nu} \left\|v^+\right\|^2_{1,S_2}, \tag{5.10}$$

where C is a constant independent of h and L.

PROOF. Denote

$$\begin{aligned} x &= \overline{\left\|v^+ - v^-\right\|}_{0,\Gamma_0}, \\ y &= \left\|v^+ - v^-\right\|_{0,\Gamma_0}, \\ z &= \left\|v^+\right\|_{1,S_2}, \\ w &= Ch^2L^{2\nu}. \end{aligned} \tag{5.11}$$

Equation (5.3) is written simply as $x \geq y - wz \geq 0$. We have

$$x^2 \geq (y - wz)^2 = y^2 - 2wyz + w^2 z^2. \tag{5.12}$$

Since $2wyz \leq y^2/2 + 2w^2z^2$, we obtain

$$x^2 \geq y^2 - \left(\frac{y^2}{2} + 2w^2 z^2\right) + w^2 z^2 = \frac{y^2}{2} - w^2 z^2. \tag{5.13}$$

This is the desired result (5.9) by noting (5.11). The proof for (5.10) is similar, and this completes the proof of Lemma 5.2.

Second, let us consider integration approximation for $\iint_{S_2} t$, where $t = t(x, y) = (\Delta u + f)(\Delta v + f)$. Let S_2 be divided into small triangles $\triangle_{ij}$ and small rectangles $\square_{ij}$,

$$S_2 = \left(\bigcup_{ij} \triangle_{ij}\right) \cup \left(\bigcup_{ij} \square_{ij}\right). \tag{5.14}$$

Denote by $\hat{t}_r$ the piecewise r-order interpolant of t on S_2, i.e.,

$$\hat{t}_r = P_r(x, y) = \sum_{i+j=0}^{r} a_{i,j} x^i y^j, \qquad (x, y) \in \triangle_{ij}, \tag{5.15}$$

or

$$\hat{t}_r = Q_r(x, y) = \sum_{i,j=0}^{r} a_{i,j} x^i y^j, \qquad (x, y) \in \square_{ij}, \tag{5.16}$$

where $a_{i,j}$ are the coefficients. Then, the integration rule in (3.7) can be viewed as

$$\begin{aligned} \sum_{ij} \alpha_{ij} g^2(P_{ij}) &= \widehat{\iint_{S_2}} g^2 \\ &= \widehat{\iint_{S_2}} t \\ &= \iint_{S_2} \hat{t}_r \\ &= \sum_{ij} \iint_{\triangle_{ij}} \hat{t}_r + \sum_{ij} \iint_{\square ij} \hat{t}_r. \end{aligned} \tag{5.17}$$

The partition in (5.14) is regular if $\max_{ij}(H_{ij}/\rho_{ij}) \leq C$, where H_{ij} is the maximal boundary length of $\triangle_{ij}$ and $\square_{ij}$, ρ_{ij} is the diameter of the incircle of $\triangle_{ij}$ and $\square_{ij}$, and C is constant independent of $H(= \max_{ij} H_{ij})$. The partition (5.14) is quasiuniform if $H/\min_{ij} H_{ij} \leq C$. Then we have the following lemma from the Bramble-Hilbert lemma [1].

LEMMA 5.3. *Let partition (5.14) be regular and quasiuniform. Then, the integration rule (5.17) has the error bound,*

$$\left| \iint_{S_2} t - \widehat{\iint_{S_2}} t \right| = \left| \iint_{S_2} (t - \hat{t}_r) \right| \leq CH^{r+1} |t|_{r+1,S_2}, \tag{5.18}$$

where C is a constant independent of H.

The integration rule on $\triangle_{ij}$ can be found in Strang and Fix [20], and the rule on $\square_{ij}$ can be formulated by the tensor product of the rule in one dimension, such as the Newton-Cotes rule or the Gaussian rule. The Legendre-Gauss rule given in (3.17)–(3.20) is just one of Gaussian rules with two boundary nodes fixed. For the Newton-Cotes rule, we may choose the uniform integration nodes. When $r = 1$ and 2, the popular trapezoidal and Simpson's rules are given. When v^+ in S_2 are polynomials of order L and choose $r = 2L$, the exact integration holds,

$$\widehat{\iint}_{S_2} \left(\Delta v^+\right)^2 = \iint_{S_2} \left(\Delta v^+\right)^2. \tag{5.19}$$

Below, we consider the approximate integration

$$\widehat{\iint}_{S_2} \left(\Delta v^+\right)^2 \approx \iint_{S_2} \left(\Delta v^+\right)^2, \tag{5.20}$$

by the rule with integration orders $r \le 2L - 1$. We have the following lemma.

LEMMA 5.4. *Let v^+ in S_2 be polynomials of order L, and the rule (5.17) with order $r \le 2L - 1$ be used for $\iint_{S_2}(\Delta u + f)(\Delta v + f)$. Also assume*

$$\left\|v^+\right\|_{\ell,S_2} \le CL^{(\ell-1)\nu} \left\|v^+\right\|_{1,S_2}, \qquad \ell \ge 1, \quad \forall\, v \in V_h^0, \tag{5.21}$$

where $\nu > 0$ is a constant independent of L. Then, there exists the bound,

$$\left|\left(\iint_{S_2} - \widehat{\iint}_{S_2}\right) (\Delta v)^2\right| \le CH^{r+1} L^{(r+3)\nu} \|v\|_{1,S_2}^2, \tag{5.22}$$

where H is the meshspacing of uniform integration nodes in S_2, and C is a constant independent of H and L.

PROOF. For the rule of order $r \le 2L - 1$, we have from Lemma 5.3 and (5.21),

$$\begin{aligned}
\left|\left(\iint_{S_2} - \widehat{\iint}_{S_2}\right) (\Delta v)^2\right| &\le CH^{r+1} \left|(\Delta v)^2\right|_{r+1,S_2} \\
&\le CH^{r+1} \sum_{i=0}^{r+1} |\Delta v|_{i,S_2} |\Delta v|_{r+1-i,S_2} \\
&\le CH^{r+1} \sum_{i=0}^{r+1} \|v\|_{i+2,S_2} \|v\|_{r+3-i,S_2} \\
&\le CH^{r+1} \sum_{i=0}^{r+1} \left(L^{(i+1)\nu} \|v\|_{1,S_2}\right) \left(L^{(r+2-i)\nu} \|v\|_{1,S_2}\right) \\
&\le CH^{r+1} L^{(r+3)\nu} \|v\|_{1,S_2}^2 .
\end{aligned} \tag{5.23}$$

This completes the proof of Lemma 5.4. ∎

THEOREM 5.1. *Let (5.2) and all conditions of Theorem 4.1 and Lemma 5.4 hold. Suppose*

$$hL^{2\nu} = o(1), \tag{5.24}$$

$$HL^{(1+2/(r+1))\nu} = o(1). \tag{5.25}$$

Then the uniformly V_h^0-elliptic inequality (3.31) holds.

PROOF. From Lemmas 5.2 and 5.4 and Theorem 4.1, we have

$$\begin{aligned}
\hat{a}(v,v) &= \iint_{S_1} |\nabla v^-|^2 + \int_{\Gamma_0} v_n^- v^- + P_c \widehat{\iint_{S_2}} (\Delta v^+)^2 \\
&\quad + \frac{P_c}{h}\overline{\|v^+ - v^-\|_{0,\Gamma_0}^2} + P_c \overline{\|v_n^+ - v_n^-\|_{0,\Gamma_0}^2} \\
&\ge \iint_{S_1} |\nabla v^-|^2 + \int_{\Gamma_0} v_n^- v^- + P_c \iint_{S_2} (\Delta v^+)^2 \\
&\quad + \frac{P_c}{2h}\|v^+ - v^-\|_{0,\Gamma_0}^2 + \frac{P_c}{2}\|v_n^+ - v_n^-\|_{0,\Gamma_0}^2 \\
&\quad - CP_c\left(h^3L^{4\nu} + h^2L^{4\nu}\right)\|v\|_{1,S_2}^2 - CH^{r+1}L^{(r+3)\nu}\|v\|_{1,S_2}^2 \\
&\ge \frac{1}{2}a(v,v) - C\left\{P_c\left(h^3L^{4\nu} + h^2L^{4\nu}\right) + H^{r+1}L^{(r+3)\nu}\right\}\|v\|_{1,S_2}^2 \\
&\ge \frac{C_0}{2}|||v|||^2 - C\left\{P_c\left(h^3L^{4\nu} + h^2L^{4\nu}\right) + H^{r+1}L^{(r+3)\nu}\right\}\|v\|_{1,S_2}^2 \\
&\ge \frac{C_0}{2}\left\{\|v\|_{1,S_1}^2 + \left\{1 - 2\frac{C}{C_0}\left[P_C\left(h^3L^{4\nu} + h^2L^{4\nu}\right) + H^{r+1}L^{(r+3)\nu}\right]\right\}\|v\|_{1,S_2}^2\right. \\
&\quad \left. + P_c\|\Delta v\|_{0,S_2}^2 + \frac{P_c}{h}\|v^+ - v^-\|_{0,\Gamma_0}^2 + P_c\|v_n^+ - v_n^-\|_{0,\Gamma_0}^2\right\} \\
&\ge \frac{C_0}{4}|||v|||^2,
\end{aligned} \tag{5.26}$$

provided that

$$2\frac{C}{C_0}\left[P_C\left(h^3L^{4\nu} + h^2L^{4\nu}\right) + H^{r+1}L^{(r+3)\nu}\right] \le \frac{1}{2}, \tag{5.27}$$

which is satisfied by (5.24) and (5.25). This completes the proof of Theorem 5.1.

When there is no approximation for $\iint_{S_2} \Delta u^+ \Delta v^+$, we have the following corollary.

COROLLARY 5.1. *Let (5.2) and all conditions of Theorem 4.1 hold. Also let the integration (5.19) in S_2 be exact. Suppose*

$$hL^{2\nu} = o(1). \tag{5.28}$$

Then, the uniformly V_h^0- elliptic inequality (3.31) holds.

Corollary 5.1 holds for the case that v^+ in S_2 are polynomials of order $L(> k)$, and that the Legendre-Gauss rule in (3.17) with $n = L-1$ is used for $\widehat{\iint_{S_2}} \Delta^2 v^+$. Next, let us consider a special case: The functions v^+ in S_2 are chosen to be the particular solutions satisfying $-\Delta v^+ = f$ in S_2 exactly. The combination of FEM-CM in (3.4) is given by

$$\hat{a}^*(\hat{u}_h, v) = f_1(v), \qquad \forall v \in V_h^0, \tag{5.29}$$

where

$$\hat{a}^*(u,v) = \iint_{S_1} \nabla u \nabla v + \int_{\Gamma_0} u_n^- v^- + \frac{P_c}{h} \widehat{\int}_{\Gamma_0} (u^+ - u^-)(v^+ - v^-) + P_c \widehat{\int}_{\Gamma_0} (u_n^+ - u_n^-)(v_n^+ - v_n^-), \tag{5.30}$$

$$f_1(v) = \iint_{S_1} fv. \tag{5.31}$$

Note that the term, $P_c \iint_{S_2} (\Delta v)^2$, disappears in computation. Obviously, Corollary 5.1 is valid for Motz's problem discussed in Section 6.2.

REMARK 5.1. Different integration rules for $\iint_{S_2} (\Delta u + f)(\Delta v + f)$ do not influence upon errors of the solutions by combinations of FEM-CM, but guarantee the uniformly V_h^0-elliptic inequality (3.31), as long as H is chosen so small to satisfy (5.25), e.g., as long as the number of collocation nodes P_{ij} in quasiuniform distribution is large enough. This conclusion is a great distinctive feature from that in the conventional analysis of FEMs.

REMARK 5.2. For Theorems 4.1 and 5.1, three inverse inequalities, equations (4.3), (5.2), and (5.21), are needed for a polygon S_2. For polynomials v^+ of order L, equation (4.3) holds for $\nu = 2$ in [6]. The proof of (5.2) and (5.21) is given in [22].

REMARK 5.3. Equations (3.12)–(3.14) represent the generalized collocation equations using other admissible functions, such as radial basis functions, the Sinc functions, etc. The analysis of this paper holds provided that the inverse inequalities (4.3), (5.2), and (5.21) are satisfied. In fact, these inequalities can be proved for radial basis functions, the Sinc functions, etc. Details of analysis and numerical examples appear in [23].

6. NUMERICAL EXPERIMENTS

6.1. Poisson's Problem

Consider Poisson's equation,

$$-\Delta u = 2\pi^2 \sin(\pi x) \cos(\pi y), \qquad \text{in } S, \tag{6.1}$$

where $S = \{(x,y) | -1 < x < 1,\ 0 < y < 1\}$, with the following Dirichlet conditions:

$$\begin{aligned} u &= 0 && \text{on } x = \pm 1 \wedge 0 \le y \le 1, \\ u &= -\sin(\pi x) && \text{on } y = 1 \wedge -1 \le x \le 1, \\ u &= \sin(\pi x) && \text{on } y = 0 \wedge -1 \le x \le 1. \end{aligned} \tag{6.2}$$

The exact solution is $u(x,y) = \sin(\pi x)\cos(\pi y)$. Divide S by Γ_0 into S_1 and S_2. The subdomain S_1 again is split into uniform regular triangular elements: $S_1 = \cup_{ij} \triangle_{ij}$, shown in Figure 2. The admissible functions are chosen as

$$v = \begin{cases} v^- = v_1, & \text{in } S_1, \\ v^+ = \sum\limits_{i,j=0}^{L} d_{ij} T_i(2x-1) T_j(2y-1), & \text{in } S_2, \end{cases} \tag{6.3}$$

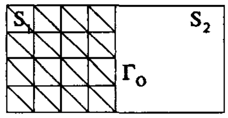

Figure 2. Partition of a rectangular solution domain with $M = 4$, where M denotes the numbers of partitions along the y-direction in S_1.

Table 1. The error norms and condition numbers by combination of FEM-CM using the Legendre-Gauss points as collocation nodes.

$M,\ L,\ N_d$	4, 3, 2	8, 5, 4	16, 7, 6
$\|u - u_h^-\|_{0,S_1}$	4.06(−2)	1.15(−2)	2.92(−3)
$\|u - u_h^-\|_{1,S_1}$	5.87(−1)	2.53(−1)	1.26(−1)
$\|u - u_L\|_{0,S_2}$	4.17(−2)	1.29(−3)	2.05(−4)
$\|u - u_L\|_{1,S_2}$	1.42(−1)	5.83(−3)	9.67(−4)
$\|\varepsilon^+ - \varepsilon^-\|_{0,\Gamma_0}$	1.92(−2)	1.80(−3)	4.83(−4)
Cond. (**A**)	8.59(4)	1.63(6)	1.22(7)

Table 2. The error norms and condition numbers by combination of FEM-CM using the trapezoidal points as collocation nodes.

$M,\ L,\ N_d$	4, 3, 4	8, 5, 6	16, 7, 8
$\|u - u_h^-\|_{0,S_1}$	6.92(−2)	1.15(−2)	2.92(−3)
$\|u - u_h^-\|_{1,S_1}$	7.88(−1)	2.53(−1)	1.26(−1)
$\|u - u_L\|_{0,S_2}$	1.64(−2)	4.27(−4)	2.23(−4)
$\|u - u_L\|_{1,S_2}$	1.09(−1)	4.24(−3)	1.04(−3)
$\|\varepsilon^+ - \varepsilon^-\|_{0,\Gamma_0}$	2.71(−2)	2.81(−3)	5.12(−4)
Cond. (**A**)	8.59(4)	1.54(6)	1.16(7)

where v_1 is the piecewise linear functions on S_1, d_{ij} are unknown coefficients to be determined, and $T_i(x)$ are the Chebyshev polynomials, $T_k(x) = \cos(k\cos^{-1}(x))$.

We choose the Legendre-Gauss and the trapezoidal rules in Section 3 for $\widehat{\iint}_{S_2}(\Delta v^+)^2$. Hence, the optimal convergence rate $O(h)$ in H^1 norms is obtained based on the analysis made. Since v^+ do not satisfy the boundary conditions on $\partial S_2 \cap \Gamma$, the additional collocation equations, $v^+(P_i) = 0$ where $P_i \in \partial S_2 \cap \Gamma$, are also needed. After trial computation, choose $P_c = 50$. Let $h = 1/M$, where M denotes the number of partitions along the y-direction in S_1 in Figure 2.
We choose Method I in Section 3, and the error norms are listed in Tables 1 and 2, where N_d denotes the number of collocation nodes along one direction in S_2 in Figure 2, $\varepsilon^- = u - u_h$ and $\varepsilon^+ = u - u_L$. The following asymptotic relations are observed from Tables 1 and 2,

$$\|u - u_h\|_{0,S_1} = O(h^2), \qquad \|u - u_h\|_{1,S_1} = O(h), \tag{6.4}$$

$$\|u - u_L\|_{0,S_2} = O(h^3), \qquad \|u - u_L\|_{1,S_2} = O(h^3), \tag{6.5}$$

$$\|\varepsilon^+ - \varepsilon^-\|_{0,\Gamma_0} = O(h^2), \qquad \text{Cond.}(\mathbf{A}) = O(h^{-3}). \tag{6.6}$$

Equations (6.4),(6.5) indicate that the numerical solutions have the optimal convergence rate $O(h)$ in H^1 norms. The different integration rules used in S_2 do not influence upon the errors

Table 3. The error norms and condition numbers by combination of FEM-CM using the trapezoidal rule with $M = 16$ and $L = 7$.

N_d	2	4	6	8	10
$\|u - u_h^-\|_{0,S_1}$	2.91(−3)	2.91(−3)	2.92(−3)	2.92(−3)	2.93(−3)
$\|u - u_h^-\|_{1,S_1}$	1.26(−1)	1.26(−1)	1.26(−1)	1.26(−1)	1.26(−1)
$\|u - u_L\|_{0,S_2}$	46.5	6.80	2.47(−4)	2.23(−4)	2.11(−4)
$\|u - u_L\|_{1,S_2}$	244.0	52.3	1.13(−3)	1.04(−3)	9.86(−4)
$\|\varepsilon^+ - \varepsilon^-\|_{0,\Gamma_0}$	28.9	4.25	5.69(−4)	5.12(−4)	4.70(−4)
Cond. (**A**)	1.14(7)	2.87(6)	1.27(6)	7.25(5)	4.80(5)

Table 4. The error norms and condition numbers by combination of FEM-CM using the Newton-Cotes rules with different orders on $M = 16$, $L = 7$, and $N_d = 7$.

Order	$r = 1$	$r = 2$	$r = 4$	$r = 8$
$\|u - u_h^-\|_{0,S_1}$	2.91(−3)	2.91(−3)	2.92(−3)	2.92(−3)
$\|u - u_h^-\|_{1,S_1}$	1.26(−1)	1.26(−1)	1.26(−1)	1.26(−1)
$\|u - u_L\|_{0,S_2}$	1.93(−4)	2.00(−4)	1.99(−4)	1.97(−4)
$\|u - u_L\|_{1,S_2}$	1.05(−3)	1.03(−3)	1.03(−3)	1.12(−3)
$\|\varepsilon^+ - \varepsilon^-\|_{0,\Gamma_0}$	6.64(−4)	7.03(−4)	6.97(−4)	6.72(−4)
Cond. (**A**)	6.59(5)	7.69(5)	8.22(5)	2.86(6)

of the solutions of combinations of FEM-CM, as long as the number of collocation equations in quasiuniform distribution is large enough.

In Table 3, we choose $M = 16$ and $L = 7$, but change the number N_d used in the trapezoidal rule in S_2. From Table 3, we can see that good solutions can be obtained when $N_d \geq 6$; this fact perfectly verifies the conclusions in Theorem 5.1. In Table 4, we choose $M = 16$, $L = 7$, and $N_d = 7$, but use the Newton-Cotes rule with difference order r. When $r = 1$, the Newton-Cotes rule is just the trapezoidal rule. From Tables 1, 2, and 4, we can see that the different integration rules used in S_2 do not influence the optimal convergence rate, either.

6.2. Motz's Problem

Consider Motz's problem,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \text{in } S, \tag{6.7}$$

where $S = \{(x,y) | -1 < x < 1,\ 0 < y < 1\}$, with the mixed type of Dirichlet-Neumann conditions,

$$\begin{aligned}
u_x &= 0, && \text{on } x = -1 \wedge 0 \leq y \leq 1, \\
u &= 500, && \text{on } x = 1 \wedge 0 \leq y \leq 1, \\
u_y &= 0, && \text{on } y = 1 \wedge -1 \leq x \leq 1, \\
u &= 0, && \text{on } y = 0 \wedge -1 \leq x < 0, \\
u_y &= 0, && \text{on } y = 0 \wedge 0 < x \leq 1.
\end{aligned} \tag{6.8}$$

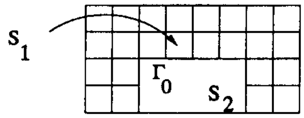

Figure 3. Partition of Motz's problem with $M = 4$.

The origin $(0, 0)$ is a singular point, since the solution behaviour $u = O(r^{1/2})$ as $r \to 0$ due to the intersection of the Dirichlet and Neumann conditions. Divide S by Γ_0 into S_1 and S_2, where $S_2 = \{(x, y) | -1/2 < x < 1/2,\ 0 < y < 1/2\}$. The subdomain S_1 is again split into uniform square elements $\square_{ij}$ with the the boundary length h, shown in Figure 3.

The admissible functions are chosen as

$$v = \begin{cases} v^- = v_1, \\ v^+ = \sum_{\ell=0}^{L} \tilde{D}_\ell r^{\ell+1/2} \cos\left(\ell + \frac{1}{2}\right)\theta, \end{cases} \tag{6.9}$$

where v_1 is the piecewise bilinear functions in S_1, $\tilde{D}_\ell$ are unknown coefficients to be determined, and (r, θ) are the polar coordinates with origin $(0, 0)$.

Since the particular solutions $r^{\ell+1/2} \cos(\ell + 1/2)\theta$ satisfy (6.7) in S_2 and the boundary conditions,

$$u = 0, \qquad \text{on } y = 0 \wedge -1 \le x < 0, \tag{6.10}$$

$$u_y = 0, \qquad \text{on } y = 0 \wedge 0 < x \le 1, \tag{6.11}$$

the collocation equations (3.9)–(3.11) are reduced to

$$\left(v^+ - v^-\right)(Q_j) = 0, \quad \left(v_n^+ - v_n^-\right)(Q_j) = 0, \qquad Q_j \in \Gamma_0. \tag{6.12}$$

Then, the collocation equations with weights on Γ_0 are given by

$$\sqrt{\frac{P_c \ell_j}{2h}} \left(v^+ - v^-\right)(Q_j) = 0, \quad \sqrt{\frac{P_c \ell_j}{2}} \left(v_n^+ - v_n^-\right)(Q_j) = 0, \qquad Q_j \in \Gamma_0, \tag{6.13}$$

where P_c is a penalty constant. In computation, we choose $P_c = 50$.

We choose Method II, where (3.26) represents (6.13). We adopt the trapezoidal and the Simpson's rules for integrals, $(P_c/h) \int_{\Gamma_0} (u^+ - u^-)(v^+ - v^-)$ and $P_c \int_{\Gamma_0} (u_n^+ - u_n^-)(v_n^+ - v_n^-)$. The error norms and the condition numbers are listed in Tables 5 and 7, where M denotes the number of partitions along the y-direction in S_1 in Figure 3, and L denotes the term number of the expansion (6.9) in S_2. Cond. denotes the condition numbers of the over-determined system (3.35). The following asymptotic relations are observed from Tables 5 and 7,

$$\|u - u_h\|_{1,S_1} + \|u - u_L\|_{1,S_2} = O(h), \tag{6.14}$$

$$\|u_I - u_h\|_{1,S_1} + \|u - u_L\|_{1,S_2} = O\left(h^{3/2-\delta}\right), \qquad 0 < \delta \ll 1, \tag{6.15}$$

$$\left\|\varepsilon^+ - \varepsilon^-\right\|_{0,\Gamma_0} = O\left(h^2\right), \qquad \text{Cond.} = O\left(h^{-1}\right). \tag{6.16}$$

We can see that equations (6.14) and (6.15) coincide with the optimal convergence rates. The approximate coefficients are given in Tables 6 and 8. When $M = 12$ and $L = 4$, the approximate

Table 5. The error norms and condition numbers for Motz's problem by combination of FEM-CM using the trapezoidal rule for $\int_{\Gamma_0}$.

$M, \quad L$	2, 2	4, 3	6, 3	8, 4	12, 4
$\|u-u_h\|_{0,S_1}$	7.45	1.32	0.960	0.342	0.231
$\|u-u_h\|_{1,S_1}$	56.6	20.9	13.9	10.3	6.92
$\|u_I-u_h\|_{0,S_1}$	6.98	1.14	0.913	0.299	0.218
$\|u_I-u_h\|_{1,S_1}$	25.6	3.51	2.53	1.01	0.767
$\|u-u_L\|_{0,S_2}$	1.32	0.424	0.358	0.0396	0.0320
$\|u-u_L\|_{1,S_2}$	4.50	1.08	0.930	0.330	0.272
$\|\varepsilon^+-\varepsilon^-\|_{0,\Gamma_0}$	6.07	1.69	0.753	0.426	0.189
Cond.	57.4	131	201	265	396

Table 6. The approximate and exact coefficients for the Motz's problem by combination of FEM-CM using the trapezoidal rule for $\int_{\Gamma_0}$.

Approx. Coeffs.	$\tilde{D}_0$	$\tilde{D}_1$	$\tilde{D}_2$	$\tilde{D}_3$	$\tilde{D}_4$
$M=2$ $L=2$	397.6500	91.0576	16.1292	/	/
$M=4$ $L=3$	399.8047	88.1314	16.8330	−7.57220	/
$M=6$ $L=3$	400.0200	88.0077	16.8798	−7.39917	/
$M=8$ $L=4$	401.1672	87.5345	16.7077	−7.49616	1.32179
$M=12$ $L=4$	401.1649	87.5318	16.8247	−7.56824	1.29397
Exact Coeffs [6]	401.1624	87.6559	17.2379	−8.07121	1.44027

value of D_0 is 401.1649, and the relative error is given by

$$\frac{|\tilde{D}_0 - D_0|}{|D_0|} = \frac{401.1649 - 401.1624}{401.1624} = 6.2 \times 10^{-6}.$$

Note that such an accuracy is higher than that given in [6], where the relative error of D_0 is about 10^{-4} when $M = 12$.

From Tables 5–8, we can see that the trapezoidal and the Simpson's rules provide almost the same results, to indicate again that different integration rules for the integrals, $(P_c/h)\int_{\Gamma_0}(u^+ - u^-)(v^+ - v^-)$ and $P_c\int_{\Gamma_0}(u_n^+ - u_n^-)(v_n^+ - v_n^-)$, do not influence upon the convergence rates of the numerical solutions if the integration nodes (i.e., the collocation nodes) are large enough. Obviously, the condition numbers given in Tables 5 and 7 are significantly smaller than those in Tables 1 and 2. Hence, Method II (i.e., the least squares method in (3.36)) is also recommended for the combinations of the collocation methods due to better numerical stability.

Table 7. The error norms and condition numbers for Motz's problem by combination of FEM-CM using the Simpson's rule for $\int_{\Gamma_0}$.

$M,\ L$	4, 3	6, 3	8, 4	12, 4
$\|u-u_h\|_{0,S_1}$	1.32	0.960	0.342	0.231
$\|u-u_h\|_{1,S_1}$	20.9	13.9	10.3	6.92
$\|u_I-u_h\|_{0,S_1}$	1.14	0.913	0.299	0.218
$\|u_I-u_h\|_{1,S_1}$	3.51	2.53	1.01	0.767
$\|u-u_L\|_{0,S_2}$	0.422	0.358	0.0396	0.0320
$\|u-u_L\|_{1,S_2}$	1.07	0.930	0.330	0.272
$\|\varepsilon^+-\varepsilon^-\|_{0,\Gamma_0}$	1.69	0.753	0.426	0.189
Cond.	165	263	350	527

Table 8. The approximate and exact coefficients for the Motz's problem by combination of FEM-CM using the Simpson's rule for $\int_{\Gamma_0}$.

Approx. Coeffs.	$\tilde{D}_0$	$\tilde{D}_1$	$\tilde{D}_2$	$\tilde{D}_3$	$\tilde{D}_4$
$M=4$ $L=3$	399.8122	88.1296	16.8320	−7.57146	/
$M=6$ $L=3$	400.0199	88.0077	16.8799	−7.39921	/
$M=8$ $L=4$	401.1671	87.5346	16.7077	−7.49609	1.32174
$M=12$ $L=4$	401.1649	87.5319	16.8247	−7.56824	1.29395
Exact Coeffs [6]	401.1624	87.6559	17.2379	−8.07121	1.44027

FINAL REMARKS

To close this paper, let us make a few remarks.

1. This paper provides a theoretical framework of combinations of CMs with other methods. The basic idea is to interpret CM as a special FEM, i.e., the LSM involving integration approximation. Equations (3.9)–(3.11) in CM are straightforwardly, and easily incorporated into the combined methods, see (3.25) and (3.26). The combination of CM in this paper is also an important development from Li [6].
2. The key analysis for combinations of CM is to prove the new uniform V_h^0- elliptic inequalities (2.17) and (3.31). The nontrivial proofs in Section 3 are new and intriguing, which consists of two steps:

Step I for the simple one (4.20) without $\int_{\Gamma_0} v_n^- v^-$;

Step II for Theorem 4.1. Note that both (2.6) and (2.7) are required in combinations (2.11) because the integral $P_c \iint_{S_2} \Delta u \Delta v$ worked as if for the biharmonic equation in S_2 in the traditional FEMs, see [1], where the essential continuity conditions $u^+ = u^-$ and $u_n^+ = u_n^-$ should be imposed on the interior boundary Γ_0.

3. In algorithms, the integration approximation leads the LSM to the collocation method. In error analysis, the integration approximation plays a role only for satisfying the uniformly V_h^0- elliptic inequality, but not for improving accuracy of the solutions. The algorithms and the analysis in this paper are distinctive from the existing literature in CM.
4. In S_2, Poisson's equation and the interior and exterior boundary conditions are copied straightforwardly into the collocation equations. This simple approach covers a large class of the CM using various admissible functions, such as particular solutions, orthogonal polynomials, the radial basis functions, the Sinc functions, see [23].
5. The numerical experiments are carried out to verify the theoretical analysis made. Also Methods I and II in Section 3 are proven to be effective in computation.

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