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Collocation methods for Poisson's equation

Hsin-Yun Hu [a], Zi-Cai Li [a,b,*]

[a] *National Center for Theoretical Sciences, Mathematics Division, National Tsing Hua University, Hsinchu 30043, Taiwan, ROC*

[b] *Department of Applied Mathematics and Department of Computer Science and Engineering, National Sun Yat-sen University, Kaohsiung 80424, Taiwan, ROC*

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Abstract

In this paper, we provide an analysis on the collocation methods (CM), which uses a large scale of admissible functions such as orthogonal polynomials, trigonometric functions, radial basis functions and particular solutions, etc. The admissible functions can be chosen to be *piecewise*, i.e., different functions are used in different subdomains. The key idea is that the collocation method can be regarded as the least squares method involving integration approximation, and optimal convergence rates can be easily achieved based on the traditional analysis of the finite element method. The key analysis is to prove the uniformly V_h-elliptic inequality and some inverse inequalities used. This paper explores the interesting fact that for the collocation methods given in this paper, the integration rules only affect on the uniformly V_h-elliptic inequality, but not on the solution accuracy. The advantage of the CM is to formulate easily the associated algebraic equations, which can be solved from the collocation equations directly by the least squares method, thus to greatly reduce the condition number of the associated matrix. Moreover, the new effective condition number is proposed to provide a better upper bound of condition number, and to show a good stability for real problems solved by the collocation methods. Note that the boundary approximation method in Li [Z.C. Li, Combined Methods for Elliptic Equations with Singularities, Interfaces and Infinities, Kluwer Academic Publishers, Boston, London, 1998] is a special case of the CM, where the admissible functions satisfy the equations exactly. Numerical experiments are also carried for Poisson's problem to support the analysis made.

AMS subject classification: 65N10; 65N30

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1. Introduction

If the admissible functions are chosen to be analytical functions, e.g., trigonometrical or other orthogonal functions, we may enforce them to satisfy directly the partial differential equations (PDEs) at certain collocation points, by letting the residuals to be zero. This leads to the collocation method (CM). The CM is a popular method in engineering computation, because the algebraic equations can be easily formulated. For CM method, there are many reference books, see in Bernardi and Maday [4], Canuto et al. [6], Gottlieb and Orszag [11], Quarteroni and Valli [20] and Mercier [18]. There have been several important studies of CM: Bernardi et al. [3], Shen [21–23], Arnold and Wendland [1], Canuto et al. [5], Pathria and Karniadakis [19], and Sneddon [24].

* Corresponding author. Address: Department of Applied Mathematics and Department of Computer Science and Engineering, National Sun Yat-sen University, Kaohsiung 80424, Taiwan, ROC. Tel.: +886 7 5252 000; fax: +886 7 5253 824.

E-mail address: zcli@math.nsysu.edu.tw (Z.-C. Li).

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In this paper, we present a new analysis of CM by following the ideas in Li [13] that other numerical methods can be regarded as a special kind of the Ritz–Galerkin method. The CM is, indeed, the least squares method, which can be treated as the Ritz–Galerkin method involving integration approximation. The advantages of the CM are twofold: (1) flexibility of application to different geometric shapes and different elliptic equations, (2) simplicity of computer programming. The optimal error bounds can be easily derived, based on the uniformly V_h-elliptic inequality which is proved in detail in this paper.

This paper is organized as follows. In the next section, the collocation method with an interior boundary is described, and in Section 3 an analysis is given. In Section 4 the CM for the Robin boundary conditions is discussed, and in Section 5, some inverse inequalities are provided. In Section 6, the numerical experiments including singularity problems are carried out to support the analysis made, and some discussions are also provided. In the last section, a few remarks are made.

2. Description of collocation methods

Consider Poisson's equation on domain S with the mixed type of the Dirichlet and Neumann conditions,

$$-\Delta u = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f(x,y) \quad \text{in } S, \tag{2.1}$$

$$u|_{\Gamma_D} = g_1 \quad \text{on } \Gamma_D, \tag{2.2}$$

$$u_\nu|_{\Gamma_N} = g_2 \quad \text{on } \Gamma_N, \tag{2.3}$$

where S is a polygon, $\partial S = \Gamma = \Gamma_D \cup \Gamma_N$ is its boundary, $u_\nu = \frac{\partial u}{\partial \nu}$, and ν is the outnormal to ∂S. Let S be divided by Γ_0 into two disjoint subregions, S_1 and S_2 (see Fig. 1): $S = S_1 \cup S_2 \cup \Gamma_0$ and $S_1 \cap S_2 = \emptyset$. We give a few assumptions.

A1: The solutions in S_1 and S_2 can be expanded as

$$v = \begin{cases} v^- = \sum_{i,j=1}^{\infty} a_{ij} \Phi_i(x)\Phi_j(y) & \text{in } S_1, \\ v^+ = \sum_{i,j=1}^{\infty} b_{ij} \Psi_i(x)\Psi_j(y) & \text{in } S_2, \end{cases} \tag{2.4}$$

where $\{\Phi_i(x)\Phi_j(y)\}$ and $\{\Psi_i(x)\Psi_j(y)\}$ are complete and independent bases in S_1 and S_2 respectively, and a_{ij} and b_{ij} are the expansion coefficients.

A2: The basis functions

$$\Phi_i(x)\Phi_j(y) \in C^2(S_1) \cap C^1(\partial S_1), \quad \Psi_i(x)\Psi_j(y) \in C^2(S_2) \cap C^1(\partial S_2). \tag{2.5}$$

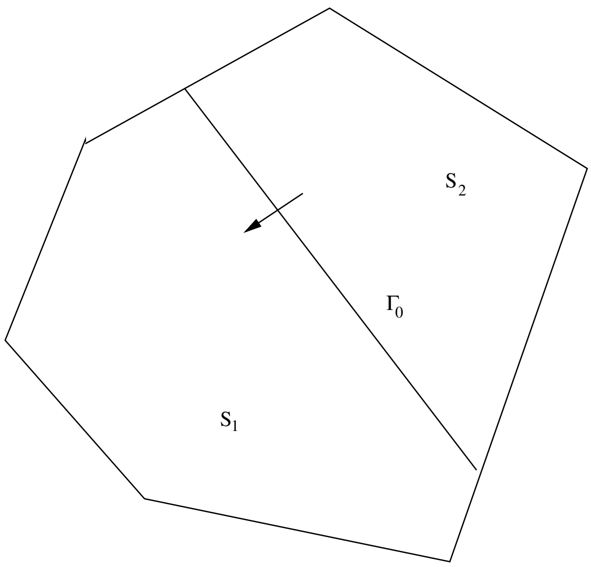

Fig. 1. Partition of a convex polygon.

A3: The expansions in (2.4) converge exponentially to the true solutions $u^{\pm}$,

$$u^- = u_m^- + R_m^-, \quad u^+ = u_n^+ + R_n^+, \tag{2.6}$$

where $u^- = u|_{S_1}$ and $u^+ = u|_{S_2}$, and

$$u_m^- = \sum_{i,j=1}^{m} a_{ij}\Phi_i(x)\Phi_j(y), \quad u_n^+ = \sum_{i,j=1}^{n} b_{ij}\Psi_i(x)\Psi_j(y), \tag{2.7}$$

R_m^- and R_n^+ are the remainders, and a_{ij} and b_{ij} are the true expansion coefficients. Then

$$\max_{S_1}|R_m^-| = O(\mathrm{e}^{-\bar{c}m}), \quad \max_{S_2}|R_n^+| = O(\mathrm{e}^{-\bar{c}n}), \tag{2.8}$$

where $\bar{c} > 0, m > 1$ and $n > 1$.

Based on **A1–A3** we may choose the *piecewise* admissible functions,

$$v = \begin{cases} v^- = \displaystyle\sum_{i,j=1}^{m} \tilde{a}_{ij}\Phi_i(x)\Phi_j(y) & \text{in } S_1, \\ v^+ = \displaystyle\sum_{i,j=1}^{n} \tilde{b}_{ij}\Psi_i(x)\Psi_j(y) & \text{in } S_2, \end{cases} \tag{2.9}$$

where $\tilde{a}_{ij}$ and $\tilde{b}_{ij}$ are unknown coefficients to be sought. Since v on Γ_0 are not continuous, $v^{\pm}$ have to satisfy the interior continuity conditions

$$u^+ = u^-, \quad u_\nu^+ = u_\nu^-, \quad \Gamma_0, \tag{2.10}$$

where $u_\nu = \frac{\partial u}{\partial \nu}$, and ν is the outward unit normal of ∂S_2.

Based on **A2** we may seek the coefficients $\tilde{a}_{ij}$ and $\tilde{b}_{ij}$ by satisfying (2.1)–(2.3) and (2.10) directly at nodes Q_{ij} and Q_i,

$$(\Delta v^{\pm} + f)(Q_{ij}^{\pm}) = 0, \quad Q_{ij}^{\pm} \in S^{\pm}, \tag{2.11}$$

$$(v - g_1)(Q_i) = 0, \quad Q_i \in \Gamma_D, \tag{2.12}$$

$$(v_\nu - g_2)(Q_i) = 0, \quad Q_i \in \Gamma_N, \tag{2.13}$$

$$(v^+ - v^-)(Q_i) = 0, \quad Q_i \in \Gamma_0, \tag{2.14}$$

$$(v_\nu^+ - v_\nu^-)(Q_i) = 0, \quad Q_i \in \Gamma_0, \tag{2.15}$$

where $v^{\pm}$ also denotes $v^{\pm}|_{\Gamma_0}$, $S^- = S_1$ and $S^+ = S_2$. Eqs. (2.11)–(2.15) can be written in a matrix form

$$\mathbf{F}\vec{\mathbf{x}} = \vec{\mathbf{b}}, \tag{2.16}$$

where $\vec{\mathbf{x}}$ is the unknown vector consisting of $\tilde{a}_{ij}$ and $\tilde{b}_{ij}$, $\vec{\mathbf{b}}$ is the known vector, and $\mathbf{F} \in R^{M\times(m^2+n^2)}$, where M ($\geqslant m^2 + n^2$) is the total number of all collocation nodes $Q_{ij}^{\pm} \in S^{\pm}$, and $Q_i \in \Gamma_D \cup \Gamma_N \cup \Gamma_0$. In this paper, we always choose $M > (m^2 + n^2)$ and even $M \gg (m^2 + n^2)$. Consequently, we obtain an overdetermined system which can be solved by the least square method (i.e., the QR decomposition or the singular value decomposition), see Golub and Loan [10].

Below, let us view the CM as the least squares method involving integration approximation. Denote by V_h the finite dimensional collection of the admissible functions (2.9). We give one more assumption.

A4: Suppose that there exists a positive constant μ (>0) such that

$$\|v_\nu^{\pm}\|_{0,\Gamma_D\cap S^{\pm}} \leqslant CL^{\mu}\|v^{\pm}\|_{1,S^{\pm}}, \quad v \in V_h, \tag{2.17}$$

$$\|v_\nu^{+}\|_{0,\Gamma_0\cap S^{+}} \leqslant CL^{\mu}\|v^{+}\|_{1,S^{+}}, \quad v \in V_h. \tag{2.18}$$

For polynomials v of order L, we will prove (2.17) and (2.18) with $\mu = 2$ in Section 5.

Then the approximate coefficients $\tilde{a}_{ij}$ and $\tilde{b}_{ij}$ can be obtained by the least squares methods: To seek the approximation solution $u_{m,n} \in V_h$ such that

$$E(u_{m,n}) = \min_{v\in V_h} E(v), \tag{2.19}$$

where

$$E(v) = \frac{1}{2}\left\{\int\int_{S_1}(\Delta v^- + f)^2 + \int\int_{S_2}(\Delta v^+ + f)^2 + L^{2\mu}\int_{\Gamma_0}(v^+ - v^-)^2 + \int_{\Gamma_0}(v_\nu^+ - v_\nu^-)^2 + L^{2\mu}\int_{\Gamma_D}(v - g_1)^2 + \int_{\Gamma_N}(v_\nu - g_2)^2\right\}, \tag{2.20}$$

where $L = \max\{m, n\}$, and ν is also the outward unit normal to ∂S_2. Eq. (2.20) can be described equivalently

$$a(u_{m,n}, v) = f(v), \quad \forall v \in V_h, \tag{2.21}$$

where

$$a(u, v) = \int\int_{S_1} \Delta u \Delta v + \int\int_{S_2} \Delta u \Delta v + L^{2\mu} \int_{\Gamma_0} (u^+ - u^-)(v^+ - v^-) + \int_{\Gamma_0} (u_\nu^+ - u_\nu^-)(v_\nu^+ - v_\nu^-) + L^{2\mu} \int_{\Gamma_D} uv + \int_{\Gamma_N} u_\nu v_\nu, \tag{2.22}$$

$$f(v) = -\int\int_{S_1} f \Delta v - \int\int_{S_2} f \Delta v + L^{2\mu} \int_{\Gamma_D} g_1 v + \int_{\Gamma_N} g_2 v_\nu. \tag{2.23}$$

The integrals in (2.22) can be approximated by some rules of integration:

$$\begin{aligned} \widehat{\int\int}_{S^\pm} g &= \sum_{ij} \alpha_{ij}^\pm g(Q_{ij}^\pm), \quad Q_{ij}^\pm \in S^\pm, \\ \widehat{\int}_{\Gamma_0} g &= \sum_i \alpha_i g(Q_i), \quad Q_i \in \Gamma_0, \\ \widehat{\int}_{\Gamma_D} g &= \sum_i \alpha_i^D g(Q_i), \quad Q_i \in \Gamma_D, \\ \widehat{\int}_{\Gamma_N} g &= \sum_i \alpha_i^N g(Q_i), \quad Q_i \in \Gamma_N, \end{aligned} \tag{2.24}$$

where $\alpha_{ij}^\pm$, α_i, α_i^D and α_i^N are positive weights, and $Q_{ij}^\pm$ and Q_i are integration nodes. The least squares method (2.21) is then reduced to

$$\hat{a}(\hat{u}_{m,n}, v) = \hat{f}(v), \quad \forall v \in V_h, \tag{2.25}$$

where

$$\hat{a}(u, v) = \widehat{\int\int}_{S_1} \Delta u \Delta v + \widehat{\int\int}_{S_2} \Delta u \Delta v + L^{2\mu} \widehat{\int}_{\Gamma_0} (u^+ - u^-)(v^+ - v^-) + \widehat{\int}_{\Gamma_0} (u_\nu^+ - u_\nu^-)(v_\nu^+ - v_\nu^-) + L^{2\mu} \widehat{\int}_{\Gamma_D} uv + \widehat{\int}_{\Gamma_N} u_\nu v_\nu, \tag{2.26}$$

$$\hat{f}(v) = -\widehat{\int\int}_{S_1} f \Delta v - \widehat{\int\int}_{S_2} f \Delta v + L^{2\mu} \widehat{\int}_{\Gamma_D} g_1 v + \widehat{\int}_{\Gamma_N} g_2 v_\nu. \tag{2.27}$$

We can see that by the rules (2.24), the following algebraic equations can be obtained from (2.25) directly,

$$\sqrt{\alpha_{ij}^\pm} (\Delta v^\pm + f)(Q_{ij}^\pm) = 0, \quad Q_{ij}^\pm \in S^\pm, \tag{2.28}$$

$$\sqrt{\alpha_i} L^\mu (v^+ - v^-)(Q_i) = 0, \quad Q_i \in \Gamma_0, \tag{2.29}$$

$$\sqrt{\alpha_i} (v_\nu^+ - v_\nu^-)(Q_i) = 0, \quad Q_i \in \Gamma_0, \tag{2.30}$$

$$\sqrt{\alpha_i^D} L^\mu (v - g_1)(Q_i) = 0, \quad Q_i \in \Gamma_D, \tag{2.31}$$

$$\sqrt{\alpha_i^N} (v_\nu - g_2)(Q_i) = 0, \quad Q_i \in \Gamma_N. \tag{2.32}$$

Compared with (2.11)–(2.15), Eqs. (2.28)–(3.32) can be denoted by

$$\mathbf{WF}\vec{\mathbf{x}} = \mathbf{W}\vec{\mathbf{b}}, \tag{2.33}$$

where $\mathbf{F}$ is given in (2.16), and $\mathbf{W} \in R^{M \times M}$ is the diagonal weight matrix, consisting of the weights, $\sqrt{\alpha_{ij}^\pm}$, $L^\mu \sqrt{\alpha_i}$, $\sqrt{\alpha_i}$, $L^\mu \sqrt{\alpha_i^D}$ and $\sqrt{\alpha_i^N}$. We may also obtain the coefficients (i.e., $\vec{x}$) by solving the normal equations:[1]

$$\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}^*, \tag{2.34}$$

[1] Eqs. (2.34) and (2.21) are used only for error analysis in this paper.

where matrix $\mathbf{A} = \mathbf{F}^T\mathbf{W}^T\mathbf{W}\mathbf{F}$ is symmetric and positive definite, and the known vector $\vec{\mathbf{b}}^* = \mathbf{F}^T\mathbf{W}^T\mathbf{W}\vec{\mathbf{b}}$. In real computation, when $M > (m^2 + n^2)$, we always solve (2.33) directly by the least squares method using the QR method, and the condition number is defined in [10] by

$$\text{Cond.} = \left\{\frac{\lambda_{\max}(\mathbf{A})}{\lambda_{\min}(\mathbf{A})}\right\}^{\frac{1}{2}}, \tag{2.35}$$

where $\lambda_{\max}(\mathbf{A})$ and $\lambda_{\min}(\mathbf{A})$ are the maximal and the minimal eigenvalues of $\mathbf{A}$, respectively. Note that Eq. (2.35) is square root of the condition number by solving the normal equation (2.34). A new effective condition number is proposed in Section 6.4 later.

3. Error analysis

We will provide the error bounds for the solutions from (2.21)–(2.25). Denote the space

$$H^* = \{v, v \in L^2(S), v^\pm \in H^1(S^\pm), \Delta v^\pm \in L^2(S^\pm)\}, \tag{3.1}$$

accompanied with the norm

$$\|v\|_H = \{\|v\|_1^2 + \|\Delta v\|_{0,S_1}^2 + \|\Delta v\|_{0,S_2}^2 + L^{2\mu}\|v^+ - v^-\|_{0,\Gamma_0}^2 + \|v_v^+ - v_v^-\|_{0,\Gamma_0}^2 + L^{2\mu}\|v\|_{0,\Gamma_D}^2 + \|v_v\|_{0,\Gamma_N}^2\}^{1/2}, \tag{3.2}$$

where

$$\|v\|_1 = \{\|v\|_{1,S_1}^2 + \|v\|_{1,S_2}^2\}^{\frac{1}{2}}, \quad |v|_1 = \{|v|_{1,S_1}^2 + |v|_{1,S_2}^2\}^{\frac{1}{2}}, \tag{3.3}$$

and $\|v\|_{1,S_1}$, $\|v\|_{0,\Gamma_0}$, etc. are the Sobolev norms. Obviously, $V_h \subset H^*$. Then

$$\|v\|_H^2 = \|v\|_1^2 + a(v, v). \tag{3.4}$$

Now we have a theorem.

Theorem 3.1. *Suppose that there exist two inequalities*

$$a(u, v) \leqslant C\|u\|_H \times \|v\|_H, \quad \forall v \in V_h, \tag{3.5}$$

$$a(v, v) \geqslant C_0\|v\|_H^2, \quad \forall v \in V_h, \tag{3.6}$$

where $C_0 > 0$ and C are two constants independent of m and n. Then, the solution of the least squares method (2.21) has the error bound,

$$\|u - u_{m,n}\|_H = C \inf_{v \in V_h} \|u - v\|_H \leqslant \varepsilon_1 = \|R_m^-\|_{2,S_1} + \|R_n^+\|_{2,S_2} + L^\mu\|R_L\|_{0,\Gamma_D\cup\Gamma_0} + \|(R_L)_v\|_{0,\Gamma_N\cup\Gamma_0}, \tag{3.7}$$

where $|R_L| = |R_m^-| + |R_n^+|$.

Proof. For the true solution, we have $a(u,v) = f(v)$, $\forall v \in V_h$. Then

$$a(u - u_{m,n}, v) = 0, \quad \forall v \in V_h. \tag{3.8}$$

Denote the projection solution on V_h

$$u_I = \begin{cases} u_I^- = \sum_{i=1}^{m} a_{ij}\Phi_i(x)\Phi_j(y) & \text{in } S_1, \\ u_I^+ = \sum_{i=1}^{n} b_{ij}\Psi_i(x)\Psi_j(y) & \text{in } S_2, \end{cases} \tag{3.9}$$

where a_{ij} and b_{ij} are the true expansion coefficients. Then $u_I \in V_h$. Let $v \in V_h$, and $w = u_{m,n} - v \in V_h$. We have from (3.6), (3.8) and (3.5)

$$C_0\|w\|_H^2 \leqslant a(u_{m,n} - v, w) = a(u - v, w) \leqslant \|u - v\|_H\|w\|_H. \tag{3.10}$$

This leads to

$$\|u_{m,n} - v\|_H = \|w\|_H \leqslant C\|u - v\|_H. \tag{3.11}$$

Then we obtain

$$\|u_{m,n} - u\|_H \leqslant \|u_{m,n} - v\|_H + \|u - v\|_H \leqslant C\|u - v\|_H \tag{3.12}$$

and

$$\|u_{m,n} - u\|_H \leqslant C \inf_{v \in V_h} \|u - v\|_H. \tag{3.13}$$

Let $v = u_I$ we have

$$\|u_{m,n} - u\|_H \leqslant C \inf_{v \in V_h} \|u - v\|_H \leqslant C\|u - u_I\|_H \leqslant \|R_m^-\|_{2,S_1} + \|R_n^+\|_{2,S_2} + L^\mu \|R_L\|_{0,\Gamma_D \cup \Gamma_0} + \|(R_L)_\nu\|_{0,\Gamma_N \cup \Gamma_0}. \tag{3.14}$$

This completes the proof of Theorem 3.1. □.

Since the solution u in (2.1)–(2.3) satisfies the finite dimensional collocation equations (2.28)–(2.32) exactly, then

$$\hat{a}(u, v) = \hat{f}(v), \quad \forall v \in V_h. \tag{3.15}$$

We can also prove the following theorem, see Ciarlet [9] and Strang and Fix [25].

Theorem 3.2. *Suppose that there exist two inequalities*

$$\hat{a}(u, v) \leqslant C\|u\|_H \times \|v\|_H, \quad \forall v \in V_h, \tag{3.16}$$

$$\hat{a}(v, v) \geqslant C_0\|v\|_H^2, \quad \forall v \in V_h, \tag{3.17}$$

where $C_0 > 0$ and C are two constants independent of m and n. Then, the solution of the collocation method (2.25) has the error bound,

$$\|u - \hat{u}_{m,n}\|_H = C \inf_{v \in V_h} \|u - v\|_H \leqslant \varepsilon_1, \tag{3.18}$$

where ε_1 is given in (3.7).

Note that for the FEM, FDM, etc., the true solution does not satisfy (3.15), then Theorem 3.2 may not hold. Also, the analysis in this paper is different form the traditional analysis in collocation method in [4,6,11,20], where only the zeros of polynomials are used as the collocation nodes.

Below we prove the uniformly V_h-elliptic inequalities (3.6) and (3.17). We cite a lemma from [14].

Lemma 3.1. *Let $\Gamma_D \cap S^- \neq \emptyset$. If $v \in H^*$, then there exists a positive constant C independent of v such that*

$$\|v\|_1 \leqslant C\{|v|_1 + \|v\|_{0,\Gamma_D} + \|v^+ - v^-\|_{0,\Gamma_0}\}.$$

Lemma 3.2. *Let* **A4** *be given, and $\Gamma_D \cap S_1 \neq \emptyset$. There exists the bound for $v \in V_h$,*

$$C_0\|v\|_1^2 \leqslant a(v, v), \quad v \in V_h, \tag{3.19}$$

where C_0 (>0) is a constant independent of m and n.

Proof. We have

$$\begin{aligned} |v|_1^2 &= \int\int_{S_1} |\nabla v|^2 + \int\int_{S_2} |\nabla v|^2 = -\int\int_{S_1} v\Delta v - \int\int_{S_2} v\Delta v + \int_{\partial S_1} v_\nu^- v^- + \int_{\partial S_2} v_\nu^+ v^+ \\ &= -\int\int_{S_1} v\Delta v - \int\int_{S_2} v\Delta v + \int_{\Gamma_0} (v_\nu^+ v^+ - v_\nu^- v^-) + \int_{\Gamma_D} v_\nu v + \int_{\Gamma_N} v_\nu v, \end{aligned} \tag{3.20}$$

where ν is the unit outnormal to ∂S or ∂S_2. Below, we give the bounds of all terms in the right hand side in the above equation.

First we have from (2.18)

$$\begin{aligned} \left|\int_{\Gamma_0} (v_\nu^+ v^+ - v_\nu^- v^-)\right| &\leqslant \left|\int_{\Gamma_0} (v^+ - v^-) v_\nu^+\right| + \left|\int_{\Gamma_0} (v_\nu^+ - v_\nu^-) v^-\right| \\ &\leqslant \|v^+ - v^-\|_{0,\Gamma_0} \|v_\nu^+\|_{0,\Gamma_0} + \|v_\nu^+ - v_\nu^-\|_{0,\Gamma_0} \|v^-\|_{0,\Gamma_0} \\ &\leqslant C\{L^\mu \|v^+ - v^-\|_{0,\Gamma_0} + \|v_\nu^+ - v_\nu^-\|_{0,\Gamma_0}\}\|v\|_1, \end{aligned} \tag{3.21}$$

where we have used the bounds,

$$\|v^-\|_{0,\Gamma_0} \leqslant \|v^-\|_{1,S^-}, \quad \|v^\pm\|_{0,S^\pm} \leqslant \|v\|_1.$$

Next, we obtain from (2.17) and (2.18)

$$\left|\int_{\Gamma_D} v_\nu v\right| \leqslant \|v\|_{0,\Gamma_D}\|v_\nu\|_{0,\Gamma_D} \leqslant CL^\mu\|v\|_{0,\Gamma_D}\|v\|_1, \tag{3.22}$$

$$\left|\int_{\Gamma_N} v_\nu v\right| \leqslant \|v_\nu\|_{0,\Gamma_N}\|v\|_{0,\Gamma_N} \leqslant C\|v_\nu\|_{0,\Gamma_N}\|v\|_1. \tag{3.23}$$

Moreover, there exist the bounds,

$$\left|\int\int_{S_1} v\Delta v\right| \leqslant \|\Delta v\|_{0,S_1}\|v\|_{0,S_1} \leqslant \|\Delta v\|_{0,S_1}\|v\|_1, \tag{3.24}$$

$$\left|\int\int_{S_2} v\Delta v\right| \leqslant \|\Delta v\|_{0,S_2}\|v\|_1. \tag{3.25}$$

From (3.19)–(3.25),

$$|v|_1^2 \leqslant \{\|\Delta v\|_{0,S_1} + \|\Delta v\|_{0,S_2} + CL^\mu(\|v^+ - v^-\|_{0,\Gamma_0} + \|v\|_{0,\Gamma_D}) + \|v_\nu^+ - v_\nu^-\|_{0,\Gamma_0} + \|v_\nu\|_{0,\Gamma_N}\}\|v\|_1. \tag{3.26}$$

Hence we have from Lemma 3.1

$$\|v\|_1^2 \leqslant C\{\|v\|_1^2 + \|v\|_{0,\Gamma_D}^2 + \|v^+ - v^-\|_{0,\Gamma_0}^2\} \leqslant C\{|v|_1^2 + (\|v\|_{0,\Gamma_D} + \|v^+ - v^-\|_{0,\Gamma_0})\|v\|_1\}. \tag{3.27}$$

Combining (3.26) and (3.27) gives

$$\|v\|_1^2 \leqslant C\{\|\Delta v\|_{0,S_1} + \|\Delta v\|_{0,S_2} + L^\mu(\|v^+ - v^-\|_{0,\Gamma_0} + \|v\|_{0,\Gamma_D}) + \|v_\nu^+ - v_\nu^-\|_{0,\Gamma_0} + \|v_\nu\|_{0,\Gamma_N}\}\|v\|_1. \tag{3.28}$$

This leads to

$$\|v\|_1 \leqslant C\{\|\Delta v\|_{0,S_1} + \|\Delta v\|_{0,S_2} + L^\mu(\|v^+ - v^-\|_{0,\Gamma_0} + \|v\|_{0,\Gamma_D}) + \|v_\nu^+ - v_\nu^-\|_{0,\Gamma_0} + \|v_\nu\|_{0,\Gamma_N}\}, \tag{3.29}$$

and then

$$\|v\|_1^2 \leqslant C\{\|\Delta v\|_{0,S_1}^2 + \|\Delta v\|_{0,S_2}^2 + L^{2\mu}(\|v^+ - v^-\|_{0,\Gamma_0}^2 + \|v\|_{0,\Gamma_D}^2) + \|v_\nu^+ - v_\nu^-\|_{0,\Gamma_0}^2 + \|v_\nu\|_{0,\Gamma_N}^2\} \leqslant Ca(v,v). \tag{3.30}$$

This is the desired result (3.19), and completes the proof of Lemma 3.2. □

Theorem 3.3. *Let* **A4** *and $\Gamma_D \cap \partial S_1 \neq \emptyset$ hold. Then there exists the uniformly V_h-elliptic inequality (3.6).*

Proof. From Lemma 3.2, we have the bound,

$$\begin{aligned} a(v,v) &= \tfrac{1}{2}a(v,v) + \tfrac{1}{2}a(v,v) \\ &\geqslant C_0\|v\|_1^2 + \tfrac{1}{2}\{\|\Delta v\|_{0,S_1}^2 + \|\Delta v\|_{0,S_2}^2 + L^{2\mu}(\|v^+ - v^-\|_{0,\Gamma_0}^2 + \|v\|_{0,\Gamma_D}^2) + \|v_\nu^+ - v_\nu^-\|_{0,\Gamma_0}^2 + \|v_\nu\|_{0,\Gamma_N}^2\} \\ &\geqslant \overline{C}_0\|v\|_H^2, \end{aligned} \tag{3.31}$$

where $\overline{C}_0 = \min\{\frac{1}{2}, C_0\}$. This completes the proof of Theorem 3.3. □

Next, we derive the uniformly V_h-elliptic inequality (3.17). We give a stronger assumption than **A4**.

A5: Suppose that there exists a positive constant μ (>0) such that for $v \in V_h$

$$\|v^\pm\|_{k,\Gamma_D\cap S^\pm} \leqslant CL^{k\mu}\|v^\pm\|_{0,\Gamma_D\cap S^\pm}, \tag{3.32}$$

$$\|v^\pm\|_{k,\Gamma_0} \leqslant CL^{k\mu}\|v^\pm\|_{0,\Gamma_0}, \tag{3.33}$$

$$\|v_\nu^\pm\|_{k,\Gamma_N\cap S^\pm} \leqslant CL^{(k+1)\mu}\|v^\pm\|_{1,S^\pm}, \tag{3.34}$$

$$\|v_\nu^\pm\|_{k,\Gamma_0\cap S^\pm} \leqslant CL^{(k+1)\mu}\|v^\pm\|_{1,S^\pm}, \tag{3.35}$$

where $k = 0, 1, \ldots$

We will give an analysis for the integration approximation. Take $\widehat{\int}_{\Gamma_0}(v_\nu^+ - v_\nu^-)^2$ as an example. Choose the integral rule of order r,

$$\widehat{\int}_{\Gamma_0} g = \int_{\Gamma_0} \hat{g}, \tag{3.36}$$

where $\hat{g}$ is the interpolant polynomial of order r on the partition Γ_0 with the maximal meshspacing h. Denote

$$\overline{\|v\|}^2_{0,\Gamma_0} = \widehat{\int_{\Gamma_0}} v^2. \tag{3.37}$$

We have the following lemma.

Lemma 3.3. *Let (3.35) be given. For rule (3.36) with order r, there exists the bound for $v \in V_h$,*

$$|\overline{\|v_\nu^+ - v_\nu^-\|}^2_{0,\Gamma_0} - \|v_\nu^+ - v_\nu^-\|^2_{0,\Gamma_0}| \leqslant Ch^{r+1}L^{(r+3)\mu}\|v\|_1^2. \tag{3.38}$$

Proof. Let $g = (v_\nu^+ - v_\nu^-)^2$. We have

$$|\overline{\|v_\nu^+ - v_\nu^-\|}^2_{0,\Gamma_0} - \|v_\nu^+ - v_\nu^-\|^2_{0,\Gamma_0}| = \left|\int_{\Gamma_0} (\hat{g} - g)\right| \leqslant Ch^{r+1}|g|_{r+1,\Gamma_0}, \tag{3.39}$$

where

$$|g|_{r+1,\Gamma_0} = |(v_\nu^+ - v_\nu^-)^2|_{r+1,\Gamma_0} \leqslant 2|(v_\nu^+)^2|_{r+1,\Gamma_0} + 2|(v_\nu^-)^2|_{r+1,\Gamma_0}. \tag{3.40}$$

From (3.35),

$$|(v_\nu^+)^2|_{r+1,\Gamma_0} \leqslant C\sum_{i=0}^{r+1}|v_\nu^+|_{r+1-i,\Gamma_0}|v_\nu^+|_{i,\Gamma_0} \leqslant C\sum_{i=0}^{r+1}(L^{(r-i+2)\mu}\|v\|_{1,S_2}) \times (L^{(i+1)\mu}\|v\|_{1,S_2}) \leqslant CL^{(r+3)\mu}\|v\|^2_{1,S_2}. \tag{3.41}$$

Similarly,

$$|(v_\nu^-)^2|_{r+1,\Gamma_0} = CL^{(r+3)\mu}\|v\|^2_{1,S_1}. \tag{3.42}$$

Combining (3.39), (3.41) and (3.42) gives the desired result (3.38). This completes the proof of Lemma 3.3. □

Similarly, we can prove the following lemma.

Lemma 3.4. *Let* **A5** *be given. For the rule (3.36) with order r, there exist the bounds for $v \in V_h$,*

$$\begin{aligned}
&|\overline{\|v^+ - v^-\|}^2_{0,\Gamma_0} - \|v^+ - v^-\|^2_{0,\Gamma_0}| \leqslant Ch^{r+1}L^{(r+1)\mu}\|v\|_1^2,\\
&|\overline{\|v\|}^2_{0,\Gamma_D} - \|v\|^2_{0,\Gamma_D}| \leqslant Ch^{r+1}L^{(r+1)\mu}\|v\|_1^2,\\
&|\overline{\|v_\nu\|}^2_{0,\Gamma_N} - \|v_\nu\|^2_{0,\Gamma_N}| \leqslant Ch^{r+1}L^{(r+3)\mu}\|v\|_1^2.
\end{aligned} \tag{3.43}$$

We give an essential assumption.

A6: Suppose that

$$\|v^\pm\|_{k,S^\pm} \leqslant CL^{(k-1)\mu}\|v\|_{1,S^\pm}, \quad k \geqslant 1,\ v \in V_h, \tag{3.44}$$

where μ is a constant independent of m and n. Choose the integral rule of order r in S,

$$\widehat{\iint_S} g = \int\int_S \hat{g}, \tag{3.45}$$

where $\hat{g}$ is the interpolant of polynomials of order r. We can also prove the following lemma easily.

Lemma 3.5. *Let* **A6** *be given and the rule (3.45) be chosen with order r. There exists the bound,*

$$\left|\left(\widehat{\iint_{S^\pm}} - \int\int_{S^\pm}\right)(\Delta v)^2\right| \leqslant Ch^{(r+1)}L^{(r+3)\mu}\|v\|^2_{1,S^\pm}. \tag{3.46}$$

Theorem 3.4. *Let* **A5–A6** *and $\Gamma_D \cap \partial S_1 \neq \emptyset$, hold. We choose h to satisfy*

$$L^{(r+3)\mu}h^{r+1} = o(1). \tag{3.47}$$

Then there exists the uniformly V_h-elliptic inequality (3.17).

Proof. We have from Theorem 3.3 and Lemmas 3.3–3.5,

$$\begin{aligned}\hat{a}(v,v) &\geqslant a(v,v) - CL^{(r+3)\mu}h^{r+1}\|v\|_1^2 \geqslant C_0\|v\|_H^2 - CL^{(r+3)\mu}h^{r+1}\|v\|_1^2\\ &\geqslant C_0\left\{\left(1-\frac{C}{C_0}L^{(r+3)\mu}h^{r+1}\right)\|v\|_1^2 + \|\Delta v\|_{0,S_1}^2 + \|\Delta v\|_{0,S_2}^2 + L^{2\mu}\|v^+ - v^-\|_{0,\Gamma_0}^2 + \|v_\nu^+ - v_\nu^-\|_{0,\Gamma_0}^2 + L^{2\mu}\|v\|_{0,\Gamma_D}^2 + \|v_\nu\|_{0,\Gamma_N}^2\right\}\\ &\geqslant \frac{C_0}{2}\|v\|_H^2, \end{aligned} \tag{3.48}$$

provided that

$$\frac{C}{C_0}L^{(r+3)\mu}h^{r+1} \leqslant \frac{1}{2}, \tag{3.49}$$

which is valid due to (3.47). □

4. The Robin boundary conditions

In the above sections and [12–14], only the Dirichlet and Neumann boundary conditions are discussed. In this section, we consider Poisson's equation involving the Robin boundary condition

$$-\Delta u = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f(x,y) \quad \text{in } S, \tag{4.1}$$

$$u_\nu|_{\Gamma_N} = g_1 \quad \text{on } \Gamma_N, \tag{4.2}$$

$$(u_\nu + \beta u)|_{\Gamma_R} = g_2 \quad \text{on } \Gamma_R, \tag{4.3}$$

where $\beta \geqslant \beta_0 > 0$, $\partial S = \Gamma = \Gamma_N \cup \Gamma_R$. Assume Meas$(\Gamma_R) > 0$ for the unique solution. For simplicity, let $\Gamma_0 = \emptyset$. (When $\Gamma_0 \neq \emptyset$, a similar analysis can be made easily by following Sections 2 and 3.) We also give two more assumptions.

A7: The solutions in S can be expanded as

$$v = \sum_{i=1}^{\infty} a_i \Phi_i \quad \text{in } S, \tag{4.4}$$

where Φ_i $(\in C^2(S) \cap C^1(\partial S))$ are complete and independent bases in S, and a_i are the exact expansion coefficients.

A8: The expansions in (4.4) converge exponentially to the true solutions u,

$$u = u_m + R_m, \tag{4.5}$$

where

$$u_m = \sum_{i=1}^{m} a_i \Phi_i, \quad R_m = \sum_{i=m+1}^{\infty} a_i \Phi_i, \tag{4.6}$$

and a_i are the true expansion coefficients. Then

$$\max_S |R_m| = O(\mathrm{e}^{-\bar{c}m}), \tag{4.7}$$

where $\bar{c} > 0$ and $m > 1$.

Based on **A7**–**A8** we may choose the uniform admissible functions,

$$v = \sum_{i=1}^{m} \tilde{a}_i \Phi_i \quad \text{in } S, \tag{4.8}$$

where $\tilde{a}_i$ are unknown coefficients to be sought. Denote by V_h the finite dimensional collection of the functions (4.8).

Choose the integral rules:

$$\widehat{\iint_S} g = \sum_{ij} \alpha_{ij} g(Q_{ij}), \quad Q_{ij} \in S, \tag{4.9}$$

$$\widehat{\int_{\Gamma_N}} g = \sum_i \alpha_i^N g(Q_i), \quad Q_i \in \Gamma_N, \tag{4.10}$$

$$\widehat{\int_{\Gamma_R}} g = \sum_i \alpha_i^R g(Q_i), \quad Q_i \in \Gamma_R. \tag{4.11}$$

We may seek the coefficients $\tilde{a}_i$ by satisfying Eqs. (4.1)–(4.3) directly at Q_{ij} and Q_i,

$$\sqrt{\alpha_{ij}}(\Delta v + f)(Q_{ij}) = 0, \quad Q_{ij} \in S, \tag{4.12}$$

$$\sqrt{\alpha_i^N}(v_\nu - g_1)(Q_i) = 0, \quad Q_i \in \Gamma_N, \tag{4.13}$$

$$\sqrt{\alpha_i^R}(v_\nu + \beta v - g_2)(Q_i) = 0, \quad Q_i \in \Gamma_R. \tag{4.14}$$

The collocation method described in (4.12)–(4.14) can be written as

$$\hat{b}(\hat{u}_m, v) = \hat{f}(v), \quad \forall v \in V_h, \tag{4.15}$$

where

$$\hat{b}(u, v) = \widehat{\iint_S} \Delta u \Delta v + \widehat{\int_{\Gamma_N}} u_\nu v_\nu + \widehat{\int_{\Gamma_R}} (u_\nu + \beta u)(v_\nu + \beta v), \tag{4.16}$$

$$\hat{f}(v) = -\widehat{\iint_S} f \Delta v + \widehat{\int_{\Gamma_N}} g_1 v_\nu + \widehat{\int_{\Gamma_R}} g_2(v_\nu + \beta v). \tag{4.17}$$

The corresponding least squares methods are then denoted by

$$b(u_m, v) = f(v), \quad \forall v \in V_h, \tag{4.18}$$

where

$$b(u, v) = \int\int_S \Delta u \Delta v + \int_{\Gamma_N} u_\nu v_\nu + \int_{\Gamma_R} (u_\nu + \beta u)(v_\nu + \beta v), \tag{4.19}$$

$$f(v) = -\int\int_S f \Delta v + \int_{\Gamma_N} g_1 v_\nu + \int_{\Gamma_R} g_2(v_\nu + \beta v). \tag{4.20}$$

Denote the norm

$$\|v\|_h = \{\|v\|_{1,S}^2 + \|\Delta v\|_{0,S}^2 + \|v_\nu\|_{0,\Gamma_N}^2 + \|(v_\nu + \beta v)\|_{0,\Gamma_R}^2\}^{1/2}. \tag{4.21}$$

Now we have a lemma.

Lemma 4.1. *Let* $\text{Meas}(\Gamma_R) > 0$. *There exists the uniformly* V_h*-elliptic inequality*

$$b(v, v) \geqslant C_0 \|v\|_{1,S}^2, \quad \forall v \in V_h. \tag{4.22}$$

Proof. We have

$$\begin{aligned} |v|_{1,S}^2 &= \int\int_S |\nabla v|^2 = -\int\int_S v\Delta v + \int\int_{\partial S} v_\nu v \leqslant \|\Delta v\|_{0,S}\|v\|_{0,S} + \int_{\Gamma_N} v_\nu v + \int_{\Gamma_R} (v_\nu + \beta v)v - \int_{\Gamma_R} \beta v^2 \\ &\leqslant \|\Delta v\|_{0,S}\|v\|_{0,S} + \|v_\nu\|_{0,\Gamma_N}\|v\|_{0,\Gamma_N} + \|v_\nu + \beta v\|_{0,\Gamma_R}\|v\|_{0,\Gamma_R} - \int_{\Gamma_R} \beta v^2 \\ &\leqslant \{\|\Delta v\|_{0,S} + \|v_\nu\|_{0,\Gamma_N} + \|v_\nu + \beta v\|_{0,\Gamma_R}\}\|v\|_{1,S} - \int_{\Gamma_R} \beta v^2, \end{aligned} \tag{4.23}$$

where we have used the bounds

$$\|v\|_{0,\Gamma_N} \leqslant C\|v\|_{1,S}, \quad \|v\|_{0,\Gamma_R} \leqslant C\|v\|_{1,S}. \tag{4.24}$$

This leads to

$$|v|_{1,S}^2 + \int_{\Gamma_R} \beta v^2 \leqslant \{\|\Delta v\|_{0,S} + \|v_\nu\|_{0,\Gamma_N} + \|v_\nu + \beta v\|_{0,\Gamma_R}\}\|v\|_{1,S}. \tag{4.25}$$

On the other hand, for $\mathrm{Meas}(\Gamma_R) > 0$,

$$\|v\|_{1,S}^2 \leqslant C(|v|_{1,S}^2 + \beta_0\|v\|_{0,\Gamma_R}^2). \tag{4.26}$$

Combining (4.25) and (4.26) gives

$$\|v\|_{1,S}^2 \leqslant C\left(|v|_{1,S}^2 + \int_{\Gamma_R} \beta v^2\right) \leqslant C\{\|\Delta v\|_{0,S} + \|v_\nu\|_{0,\Gamma_N} + \|v_\nu + \beta v\|_{0,\Gamma_R}\}\|v\|_{1,S}. \tag{4.27}$$

This leads to

$$\|v\|_{1,S} \leqslant C\{\|\Delta v\|_{0,S} + \|v_\nu\|_{0,\Gamma_N} + \|v_\nu + \beta v\|_{0,\Gamma_R}\}, \tag{4.28}$$

and then

$$\|v\|_{1,S}^2 \leqslant C\{\|\Delta v\|_{0,S}^2 + \|v_\nu\|_{0,\Gamma_N}^2 + \|v_\nu + \beta v\|_{0,\Gamma_R}^2\} = Cb(v,v). \tag{4.29}$$

This is (4.22) and completes the proof of Lemma 4.1. □

Note that the true solution u also satisfies (4.15) exactly,

$$\hat{b}(u,v) = \hat{f}(v), \quad \forall v \in V_h.$$

A similar argument as in Theorem 3.4 can be given for (4.31): $\hat{b}(v,v) \geqslant C_0\|v\|_h^2$, $\forall v \in V_h$. We can obtain the following theorem by following Sections 2 and 3.

Theorem 4.1. *Suppose that there exist two inequalities,*

$$\hat{b}(u,v) \leqslant C\|u\|_h \times \|v\|_h, \quad \forall v \in V_h, \tag{4.30}$$

$$\hat{b}(v,v) \geqslant C_0\|v\|_h^2, \quad \forall v \in V_h, \tag{4.31}$$

where $C_0 > 0$ and C are two constants independent of m. Then, the solution of the collocation method (4.15) has the error bound,

$$\|u - \hat{u}_m\|_h = C \inf_{v \in V_h} \|u - v\|_h \leqslant C\{\|R_m\|_{2,S} + \|(R_m)_\nu\|_{0,\Gamma_N} + \|(R_m)_\nu\|_{0,\Gamma_R}\}. \tag{4.32}$$

Note that when the admissible functions are chosen to satisfy Poisson's equation, the boundary approximation method (BAM) is then obtained from the CM. Hence the BAM is a special case of the CM in this paper. Moreover, in traditional CM, some difficulties are encountered for the Neumann boundary conditions, see [20]. In this paper, the techniques given can handle well both the Neumann and the Robin boundary conditions.

5. Inverses inequalities

In the above analysis, we need the inverse estimates in **A4**, **A5** and **A6**. In fact the inverse estimates in **A6** is essential. Take the norms on Γ_0 for example. We have from assumption **A6**

$$\|v^+\|_{k,\Gamma_0} \leqslant C\|v^+\|_{k+1,S^+} \leqslant CL^{k\mu}\|v^+\|_{1,S^+}, \tag{5.1}$$

$$\|v_\nu^+\|_{k,\Gamma_0} \leqslant C\|v^+\|_{k+2,S^+} \leqslant CL^{(k+1)\mu}\|v^+\|_{1,S^+}. \tag{5.2}$$

Hence, **A4** and **A5** can be replaced by (5.1), (5.2), etc., and the proof for Lemma 3.4 and Theorem 3.4 is similar.

To prove the inverse inequalities, in this paper we confine the smooth solution of (2.1)–(2.3), and choose admissible functions $\Phi_i(x)$ and $\Psi_i(x)$ in (2.4), and Φ_i in (4.4) as polynomials of order i, then Theorem 5.1 yields the essential inverse inequality. As to other admissible functions, such as radial basis functions, the inverse inequality is proven in Hu et al. [12]. As long as the inverse inequalities hold, the uniformly V_h-elliptic inequality holds and then the optimal error estimates can be achieved.

First we cite the results in Li [13, pp. 161–163], as two lemmas.

Lemma 5.1. *Let $\rho_L = \rho_L(x)$ be an L-order polynomial on $[-1,1]$. Then there exists a constant C independent of L such that*

$$\|\rho_L'\|_{0,[-1,1]} \leqslant CL^2\|\rho_L\|_{0,[-1,1]}. \tag{5.3}$$

Lemma 5.2. *Suppose that Γ_0 is made up of finite sections of straight lines, and that the admissible function w_h in S_2 is an L-order polynomial. Then there exists a constant C independent of L such that*

$$\sup_{w_h \in V_h} \frac{|w_h^+|_{k+1,\Gamma_0}}{\|w_h\|_1} \leqslant CL^{2(k+1)}. \tag{5.4}$$

Lemma 5.3. *Let $\square = \{(x,y),\ -1 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 1\}$, and choose*

$$w_L = \sum_{i,j=0}^{L} a_{ij} T_i(x) T_j(y), \quad (x,y) \in \square, \tag{5.5}$$

where a_{ij} are expansion coefficients, and $T_i(x)$ are the Chebyshev polynomials of order i. Then there exist the inverse inequalities,

$$\left\| \frac{\partial}{\partial x} w_L \right\|_{0,\square} \leqslant CL^2 \|w_L\|_{0,\square}, \tag{5.6}$$

$$\left\| \frac{\partial}{\partial y} w_L \right\|_{0,\square} \leqslant CL^2 \|w_L\|_{0,\square}, \tag{5.7}$$

where C is a constant independent of L.

Proof. We prove (5.6) only, since the proof for (5.7) is similar. We may express w_L by the Legendre polynomials

$$w_L = \sum_{i,j=0}^{L} b_{ij} P_i(x) P_j(y), \quad (x,y) \in \square, \tag{5.8}$$

where the coefficients b_{ij} from (5.5) are uniquely determined. We have from the orthogonality of the Legendre polynomials,

$$\|w_L\|_{0,\square}^2 = \int\int_{\square} \left(\sum_{i,j=0}^{L} b_{ij} P_i(x) P_j(y) \right)^2 = \sum_{i,j=0}^{L} \frac{4b_{ij}^2}{(2i+1)(2j+1)} = \sum_{j=0}^{L} \frac{2}{2j+1} \sum_{i=0}^{L} \frac{2b_{ij}^2}{2i+1} = \sum_{j=0}^{L} \frac{2}{2j+1} \|z_j\|_{0,[-1,1]}^2, \tag{5.9}$$

where z_j are polynomials of order L,

$$z_j = z_j(x) = \sum_{i=0}^{L} b_{ij} P_i(x), \quad x \in [-1,1]. \tag{5.10}$$

On the other hand, we have

$$\left\| \frac{\partial}{\partial x} w_L \right\|_{0,\square}^2 = \int\int_{\square} \left(\sum_{i,j=0}^{L} b_{ij}^2 P_i'(x) P_j(y) \right)^2 = \sum_{j=0}^{L} \frac{2}{2j+1} \int_{-1}^{1} \sum_{i,\bar{i}=0}^{L} b_{ij} b_{\bar{i}j} P_i'(x) P_{\bar{i}}'(x)\, dx = \sum_{j=0}^{L} \frac{2}{2j+1} \|z_j'(x)\|_{0,[-1,1]}^2, \tag{5.11}$$

where the polynomials $z_j(x)$ are given in (5.10). Based on Lemma 5.1, we have from (5.9) and (5.11)

$$\left\| \frac{\partial}{\partial x} w_L \right\|_{0,\square}^2 = \sum_{j=0}^{L} \frac{2}{2j+1} \|z_j'(x)\|_{0,[-1,1]}^2 \leqslant CL^4 \sum_{j=0}^{L} \frac{2}{2j+1} \|z_j(x)\|_{0,[-1,1]}^2 = CL^4 \sum_{i,j=0}^{L} \frac{4b_{ij}^2}{(2i+1)(2j+1)} = CL^4 \|w_L\|_{\square}^2. \tag{5.12}$$

This is the desired result (5.6) and completes the proof of Lemma 5.3. □

A9: Let S be an polygon shown in Fig. 2. Then S can be decomposed of finite quasiuniform parallelogram $\Omega_i : S = \cup_i \Omega_i$, where overlap of Ω_i is allowed, see Babuska and Guo [2]. By the diagonal line, Ω_i is split into two triangles, $\triangle_i^+$ and $\triangle_i^-$. Suppose that all $\triangle_i^\pm$ are quasiuniform, e.g., $c_0 \leqslant \rho_i^\pm$, where $\rho_i^\pm$ denotes the radius of the largest inscribed ball of $\triangle_i^\pm$, and c_0 is a positive constant. We have the following lemma.

Theorem 5.1. *Let* **A9** *be given. For the polynomial w_L in (5.5). Then there exists a constant C independent of L such that*

$$\|w_L\|_{k,S} \leqslant CL^{2k} \|w_L\|_{0,S}. \tag{5.13}$$

Proof. Consider the parallelograms Ω_i in Fig. 3, where a_i and b_i are the lengths of two edges, and α_i are the angles between Ω_i and the y-axis. From the quasiuniform parallelograms, where exist the bounds,

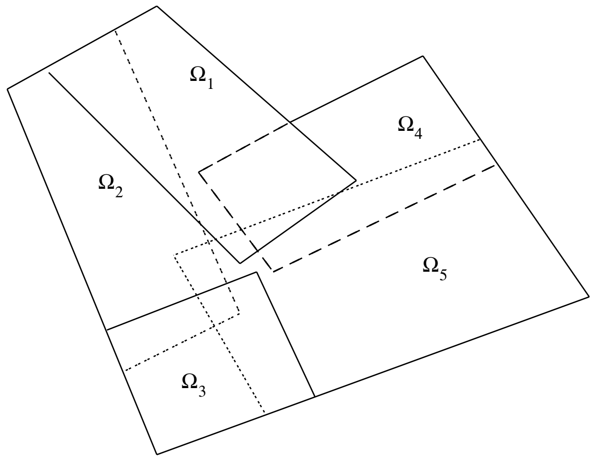

Fig. 2. A polygon decomposed of finite parallelograms Ω_i.

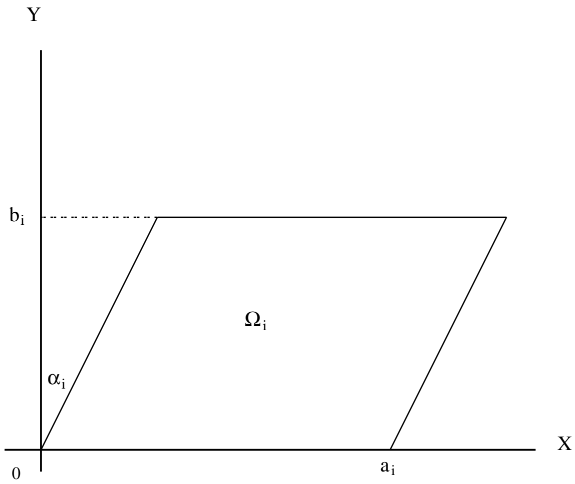

Fig. 3. A parallelogram Ω_i.

$$0 < a_i, b_i < C, \quad \frac{\max\{a_i, b_i\}}{\min\{a_i, b_i\}} \leqslant C, \quad 0 \leqslant \alpha_i \leqslant \alpha_M < \frac{\pi}{2}. \tag{5.14}$$

The parallelograms Ω_i can be transformed to $\square$ by the linear transformation $T : (x, y) \rightarrow (\hat{x}, \hat{y})$, where

$$\hat{x} = \frac{2}{a_i}(x - (\tan\alpha_i)y) - 1,$$
$$\hat{y} = \frac{2}{b_i}y - 1.$$

Denote $\hat{w} = T(w)$. We have

$$\frac{\partial w}{\partial x} = \frac{2}{a_i}\frac{\partial \hat{w}}{\partial \hat{x}},$$
$$\frac{\partial w}{\partial y} = -\frac{2}{a_i}(\tan\alpha_i)\frac{\partial \hat{w}}{\partial \hat{x}} + \frac{2}{b_i}\frac{\partial \hat{w}}{\partial \hat{y}}.$$

Through the linear transformations T, we obtain

$$|w|^2_{1,\Omega_i} = \int\int_{\Omega_i} (w_x^2 + w_y^2)\,\mathrm{d}x\,\mathrm{d}y = \frac{a_i b_i}{4}\int\int_{\square}\left\{\left(\frac{2}{a_i}\frac{\partial \hat{w}}{\partial \hat{x}}\right)^2 + \left(-\frac{2}{a_i}(\tan\alpha_i)\frac{\partial \hat{w}}{\partial \hat{x}} + \frac{2}{b_i}\frac{\partial \hat{w}}{\partial \hat{y}}\right)^2\right\}\mathrm{d}\hat{x}\,\mathrm{d}\hat{y}$$
$$\leqslant \frac{a_i b_i}{4}\int\int_{\square}\left\{\frac{4}{a_i^2}(1 + 2\tan^2\alpha_i)\left(\frac{\partial \hat{w}}{\partial \hat{x}}\right)^2 + 2\frac{4}{b_i^2}\left(\frac{\partial \hat{w}}{\partial \hat{y}}\right)^2\right\}\mathrm{d}\hat{x}\,\mathrm{d}\hat{y} \leqslant C\int\int_{\square}\left\{\left(\frac{\partial \hat{w}}{\partial \hat{x}}\right)^2 + \left(\frac{\partial \hat{w}}{\partial \hat{y}}\right)^2\right\}\mathrm{d}\hat{x}\,\mathrm{d}\hat{y}, \tag{5.15}$$

where the constant

$$C = \max_i \left\{ \frac{b_i}{a_i}(1 + 2\tan^2\alpha_i) + \frac{2a_i}{b_i} \right\}.$$

The constant C is independent of i due to assumption (5.14). Under the linear transformation T, the polynomials of order L remain as well. Based on Lemma 5.3, we have for $w = w_L$.

$$\int\int_{\square} \left\{ \left(\frac{\partial \hat{w}}{\partial \hat{x}}\right)^2 + \left(\frac{\partial \hat{w}}{\partial \hat{y}}\right)^2 \right\} \mathrm{d}\hat{x}\,\mathrm{d}\hat{y} \leqslant CL^4 \int\int_{\square} (\hat{w})^2\,\mathrm{d}\hat{x}\,\mathrm{d}\hat{y}. \tag{5.16}$$

Moreover, through the inverse transformation $\hat{T}$, we obtain for $w = w_L$

$$\int\int_{\square} (\hat{w})^2\,\mathrm{d}\hat{x}\,\mathrm{d}\hat{y} = \frac{4}{a_i b_i} \int\int_{\Omega_i} w^2\,\mathrm{d}x\,\mathrm{d}y \leqslant C\|w\|^2_{0,\Omega_i}. \tag{5.17}$$

Combining (5.15) and (5.17) gives

$$|w|_{1,\Omega_i} \leqslant CL^2\|w\|_{0,\Omega_i},$$

and

$$\|w\|_{1,\Omega_i} \leqslant CL^2\|w\|_{0,\Omega_i}.$$

Consequently, for parallelograms Ω_i we have

$$\|w_L\|_{k,\Omega_i} \leqslant C(L-k)^2\|w_L\|_{k-1,\Omega_i} \leqslant CL^2\|w_L\|_{k-1,\Omega_i} \leqslant CL^{2k}\|w_L\|_{0,\Omega_i}.$$

From **A9** we obtain

$$\|w_L\|^2_{k,S} \leqslant \sum_i \|w_L\|^2_{k,\Omega_i} \leqslant CL^{4k} \sum_i \|w_L\|^2_{0,\Omega_i} \leqslant CL^{4k}\|w_L\|^2_{0,S},$$

by noting finite overlaps of Ω_i. This completes the proof of Theorem 5.1. □

6. Numerical experiments and discussions

In the section, we carry out three computational models by using the collocation methods.

6.1. Radial basis functions for different boundary conditions

When the radial basis functions are chosen as the admissible functions, the analysis (in particular the inverse inequalities) is made in Hu et al. [12]. Here, we only provide numerical experiments, to show the effectiveness of the CM in this paper. The radial basis functions are discussed in [16,17].

First, we consider Poisson's equation on domain S with the mixed type of boundary conditions

$$-\Delta u = 2\pi^2 \sin(\pi x)\cos(\pi y), \quad \text{in } S, \tag{6.1}$$

$$u = 0 \quad \text{on } x = 1 \wedge 0 \leqslant y \leqslant 1, \tag{6.2}$$

$$u_x + 2u = -\pi\cos(\pi y) \quad \text{on } x = -1 \wedge 0 \leqslant y \leqslant 1,$$

$$u_y = 0 \quad \text{on } y = 1 \wedge -1 \leqslant x \leqslant 1,$$

$$u = \sin(\pi x) \quad \text{on } y = 0 \wedge -1 \leqslant x \leqslant 1.$$

where S is a rectangle, $S = \{(x,y) | -1 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$, ∂S is its boundary. Choose the solution

$$u(x,y) = \sin(\pi x)\cos(\pi y). \tag{6.3}$$

The purpose of this experiment is to apply the collocation methods using the radial basis functions for different boundary conditions.

The admissible functions are chosen as

$$v = \sum_{i=1}^{N_S} a_i g_i(x,y), \quad \text{in } S, \tag{6.4}$$

Table 1
The error norms and condition number by the inverse multiquadric radial basis collocation method with parameter $c = 2.0$

N_d, L	16, 4	16, 6	16, 8	16, 10
$\|u - v\|_{0,\infty,S}$	1.72	7.27(−2)	5.28(−4)	2.78(−5)
$\|u - v\|_{0,S}$	4.83(−1)	2.21(−2)	8.87(−5)	1.26(−5)
$\|u - v\|_{1,S}$	2.02	9.09(−2)	9.48(−4)	8.16(−5)
Cond.(**A**)	2.00(3)	6.36(5)	3.81(8)	1.39(9)

Table 2
The error norms and condition number by the Gaussian radial basis collocation method with parameter $c = 2.0$

N_d, L	16, 4	16, 6	16, 8	16, 10
$\|u - v\|_{0,\infty,S}$	1.53	8.25(−2)	7.46(−4)	1.37(−5)
$\|u - v\|_{0,S}$	4.80(−1)	2.43(−2)	1.60(−4)	3.44(−6)
$\|u - v\|_{1,S}$	1.92	1.15(−1)	1.30(−3)	3.65(−5)
Cond.(**A**)	1.08(4)	1.53(8)	1.05(9)	3.69(9)

where a_i are unknown coefficients to be determined, and $g_i(x,y)$ are the radial basis function. First, we choose the inverse multiquadric radial basis functions (IMQRB)

$$g_i(x,y) = \frac{1}{\sqrt{r_i^2 + c^2}}, \tag{6.5}$$

where c is a parameter constant, $r_i = \{(x - x_i)^2 + (y - y_i)^2\}^{\frac{1}{2}}$, and (x_i, y_i) are the source points which may not be necessarily chosen as the collocation nodes. Suitable additional functions may be added into (6.4), such as some polynomials and the singular functions if necessary. Next, we may choose the Gaussian radial basis functions (GRB)

$$g_i(x,y) = \exp\left(-\frac{r_i^2}{c^2}\right). \tag{6.6}$$

We use the collocation equations (4.12)–(4.14) on the uniform interior collocation nodes, and choose the trapezoidal rules for (4.9)–(4.11). The error norms are listed in Tables 1 and 2 for IMQRB and GRB, respectively, where N_d denotes the number of partition along the y-direction in S. Let the source points of radial basis functions also be the collocation nodes. And let $N_S = L^2$ in (6.4). From Table 1, we can see the following asymptotic relations for IMQRB

$$\|u - v\|_{0,\infty,S} = O((0.16)^L), \tag{6.7}$$

$$\|u - v\|_{0,S} = O((0.17)^L), \tag{6.8}$$

$$\|u - v\|_{1,S} = O((0.19)^L). \tag{6.9}$$

And from Table 2 we can see for GRB,

$$\|u - v\|_{0,\infty,S} = O((0.15)^L), \tag{6.10}$$

$$\|u - v\|_{0,S} = O((0.14)^L), \tag{6.11}$$

$$\|u - v\|_{1,S} = O((0.16)^L). \tag{6.12}$$

Eqs. (6.7)–(6.12) indicate that the numerical solutions have the exponential convergence rates. Note that the GRB collocation method converges slightly faster than the IMQRB collocation method.

6.2. Piecewise admissible functions

Next, consider Poisson's equation,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \text{in } S, \tag{6.13}$$

where $S = \{(x,y) | -1 < x < 1,\ 0 < y < 1\}$, with the following boundary conditions:

Table 3
The error norms and condition number by combination of CMs with the different expansion terms L and M used in S_1 and S_2, respectively, and the number N of collocation nodes along y-axis in a uniform distribution

L, M, N	3, 5, 6	4, 6, 8	5, 7, 10	6, 8, 12	7, 9, 14
$\|u-v^-\|_{0,\infty,S_1}$	3.41(−1)	1.39(−1)	9.23(−3)	1.86(−3)	2.29(−4)
$\|u-v^-\|_{0,S_1}$	6.92(−2)	3.75(−2)	2.12(−3)	4.54(−4)	5.31(−5)
$\|u-v^-\|_{1,S_1}$	5.23(−1)	1.85(−1)	1.69(−2)	3.82(−3)	3.33(−4)
$\|u-v^+\|_{0,\infty,S_2}$	3.09(−1)	1.46(−1)	8.15(−3)	1.83(−3)	2.06(−4)
$\|u-v^+\|_{0,S_2}$	6.26(−2)	3.59(−2)	1.91(−3)	3.59(−4)	4.72(−5)
$\|u-v^+\|_{1,S_2}$	4.74(−1)	1.74(−1)	1.43(−2)	3.30(−3)	2.86(−4)
$\|\varepsilon^+-\varepsilon^-\|_{0,\Gamma_0}$	1.23(−3)	9.34(−4)	1.56(−5)	8.59(−6)	2.64(−7)
Cond.(**A**)	7.96(2)	1.52(3)	2.82(3)	4.82(3)	7.80(3)

$$
\begin{aligned}
&u=\cos(\pi y) \quad \text{on } x=-1 \ \wedge\ 0\leqslant y\leqslant 1,\\
&u=\cosh(2\pi)\cos(\pi y) \quad \text{on } x=1 \ \wedge\ 0\leqslant y\leqslant 1,\\
&u=-\cosh((x+1)\pi) \quad \text{on } y=1 \ \wedge\ -1\leqslant x\leqslant 1,\\
&u_y=0 \quad \text{on } y=0 \ \wedge\ -1\leqslant x\leqslant 1.\\
&u^+=u^-, \quad u_\nu^+=u_\nu^- \quad \text{on } \Gamma_0.
\end{aligned}
\tag{6.14}
$$

The exact solution is $u(x,y)=\cosh(\pi(x+1))\cos(\pi y)$. Divide S by Γ_0 into S_1 and S_2, where $S_1=\{(x,y)|\ -1<x<0,\ 0<y<1\}$ and $S_2=\{(x,y)|\ 0<x<1,\ 0<y<1\}$. The admissible functions are chosen as

$$
v=\begin{cases} v^-=\sum_{i,j=0}^{L} a_{ij}T_i(2x+1)T_j(2y-1), & \text{in } S_1,\\ v^+=\sum_{i,j=0}^{M} d_{ij}T_i(2x-1)T_j(2y-1), & \text{in } S_2, \end{cases}
\tag{6.15}
$$

where a_{ij} and d_{ij} are unknown coefficients to be determined, and $T_i(x)$ are the Chebyshev polynomials $T_k(x)=\cos(k\cos^{-1}(x))$.

Then, we use the collocation equations, (2.28)–(2.32), and choose the trapezoidal rules for (2.24). Since $v^\pm$ do not satisfy the boundary conditions, some additional collocation $v^\pm(P_i)=0$, $P_i\in\partial S$, are also needed. The error norms are listed in Table 3, the different expansion terms L and M are used in S_1 and S_2, respectively, and N is the number of collocation nodes along y-axis in a uniform distribution. The following asymptotic relations are observed:

$$\|u-v^-\|_{0,\infty,S_1}=O((0.161)^L), \quad \|u-v^+\|_{0,\infty,S_2}=O((0.161)^M), \tag{6.16}$$

$$\|u-v^-\|_{0,S_1}=O((0.166)^L), \quad \|u-v^+\|_{0,S_2}=O((0.166)^M), \tag{6.17}$$

$$\|u-v^-\|_{1,S_1}=O((0.159)^L), \quad \|u-v^+\|_{1,S_2}=O((0.157)^M), \tag{6.18}$$

$$\|\varepsilon^+-\varepsilon^-\|_{0,\Gamma_0}=O((0.121)^{L_M}), \quad \text{Cond.(A)}=O((1.78)^{L_M}), \tag{6.19}$$

where $L_M=\max\{L,M\}$. Above equations indicate that the numerical solutions satisfy (2.8), to have justified the analysis in Section 3.

6.3. Motz's problem

Finally, consider Motz's problem,

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \quad \text{in } S, \tag{6.20}$$

where $S=\{(x,y)|\ -1<x<1,\ 0<y<1\}$, with the following mixed type of Dirichlet–Neumann conditions, see Fig. 4, where $u_n=\frac{\partial u}{\partial n}$, and n is the outnormal to ∂S.

$$
\begin{aligned}
&u_x=0 \quad \text{on } x=-1 \ \wedge\ 0\leqslant y\leqslant 1,\\
&u=500 \quad \text{on } x=1 \ \wedge\ 0\leqslant y\leqslant 1,\\
&u_y=0 \quad \text{on } y=1 \ \wedge\ -1\leqslant x\leqslant 1,\\
&u=0 \quad \text{on } y=0 \ \wedge\ -1\leqslant x<0,\\
&u_y=0 \quad \text{on } y=0 \ \wedge\ 0<x\leqslant 1.
\end{aligned}
\tag{6.21}
$$

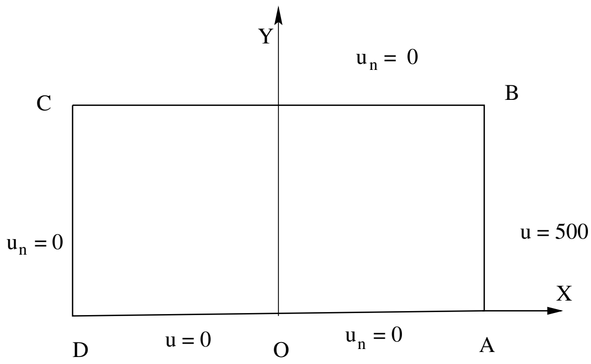

Fig. 4. Motz's problem.

The origin $(0,0)$ is a singular point, since the solution behaviour is $u = O(r^{\frac{1}{2}})$ as $r \to 0$ due to the intersection of the Dirichlet and Neumann conditions.

The admissible functions are chosen as

$$v = \sum_{\ell=0}^{L} \widetilde{D}_\ell r^{\ell+\frac{1}{2}} \cos\left(\ell + \frac{1}{2}\right)\theta, \tag{6.22}$$

where $\widetilde{D}_\ell$ are unknown coefficients to be determined, and (r,θ) are the polar coordinates with origin $(0,0)$. Note that the particular solutions $r^{\ell+\frac{1}{2}}\cos(\ell+\frac{1}{2})\theta$ satisfy (6.20) and the boundary conditions exactly:

$$u = 0 \quad \text{on } y = 0 \ \wedge \ -1 \leqslant x < 0, \tag{6.23}$$

$$u_y = 0 \quad \text{on } y = 0 \ \wedge \ 0 < x \leqslant 1. \tag{6.24}$$

The unknown coefficients $\widetilde{D}_\ell$ in (6.22) can be determined by satisfying the rest boundary conditions.

Denote by V_h the finite dimensional collocation of (6.22). Then the CM can be denoted by: To seek u_L such that

$$I(u_L) = \min_{v \in V_h} I(v), \tag{6.25}$$

where

$$I(v) = \int_{\overline{AB}} (v-500)^2 + w_1^2 \int_{\overline{BC}} v_v^2 + w_1^2 \int_{\overline{CD}} v_v^2 \tag{6.26}$$

with $w_1 = \frac{1}{L}$. Denote

$$\|\epsilon\|_B = \|u - u_L\|_B = \sqrt{I(u-u_L)} = \sqrt{I(v)}. \tag{6.27}$$

Then, we use the collocation equations only on the rest of the boundary,

$$\sqrt{\alpha_i}(v-500)(Q_i) = 0, \quad Q_i \text{ on } x = 1 \ \wedge \ 0 \leqslant y \leqslant 1, \tag{6.28}$$

$$w_1\sqrt{\alpha_i} v_v(Q_i) = 0, \quad Q_i \text{ on } x = -1 \ \wedge \ 0 \leqslant y \leqslant 1, \tag{6.29}$$

$$w_1\sqrt{\alpha_i} v_v(Q_i) = 0, \quad Q_i \text{ on } y = 1 \ \wedge \ -1 \leqslant x \leqslant 1, \tag{6.30}$$

where Q_i and α_i are the nodes and weights of some integral rules, respectively. Note that Eqs. (6.28)–(6.30) are just the boundary approximation method (BAM) in [14], which is a special case of the CM in this paper. Also note that the collocation nodes Q_i in (6.28)–(6.30) are far from the singular origin. Eqs. (6.28)–(6.30) can be denoted in the form of matrix and vectors,

$$\mathbf{F}\vec{\mathbf{x}} = \vec{\mathbf{b}}, \tag{6.31}$$

where $\mathbf{F}$ $(\in R^{m\times n}, m \geqslant n)$ is the stiffness matrix, $\vec{\mathbf{x}}$ $(\in R^n)$ is the unknown vector consisting of $\widetilde{D}_\ell$, and $\vec{\mathbf{b}}$ $(\in R^m)$ is the known vector. Eq. (6.31) is the over-determined system, and can be solved by the least squares method using the QR method.

Denote by N the collocation number of ∂S along axis Y, the total number of all collocation nodes are $4N$. First, choose the centroid rule, errors of the solutions from the BAM and condition number are listed in Tables 4 and 5. Moreover, choose the Gaussian rule with six nodes and the Gaussian rules with 1, 2, 4, 6, 8 and 10 nodes, the results are listed in Tables 6 and 7, and the best leading coefficients in Table 8 by the Gaussian rule with six nodes as $L = 34$ and $N = 30$.

For the centroid rule, the number of collocation nodes should be large enough to guarantee the uniformly V_h-elliptic inequality. It can be seen from Table 5 that N should be chosen as $N \geqslant \frac{L}{2}$ for $L = 34$. From Table 7, we also conclude that

Table 4
The error norms and condition number by the BAM with centroid rule

L	N	$\|\varepsilon\|_B$	$\|\varepsilon\|_{\infty,\overline{AB}}$	Cond.(**A**)	$\left\|\frac{\Delta D_0}{D_0}\right\|$	$\left\|\frac{\Delta D_1}{D_1}\right\|$	$\left\|\frac{\Delta D_2}{D_2}\right\|$	$\left\|\frac{\Delta D_3}{D_3}\right\|$
10	8	0.250(−1)	0.149(−1)	94.3	0.189(−5)	0.491(−5)	0.601(−5)	0.928(−3)
18	12	0.133(−3)	0.811(−4)	0.193(4)	0.158(−7)	0.113(−6)	0.290(−6)	0.502(−6)
26	16	0.973(−6)	0.734(−6)	0.366(5)	0.216(−9)	0.155(−8)	0.380(−8)	0.202(−8)
34	24	0.839(−8)	0.459(−8)	0.666(6)	0.169(−11)	0.121(−10)	0.296(−10)	0.152(−10)

Table 5
The error norms and condition number by the BAM with the centroid rule as $L = 34$

N	$\|\varepsilon\|_B$	$\|\varepsilon\|_{\infty,\overline{AB}}$	Cond.(**A**)	$\left\|\frac{\Delta D_0}{D_0}\right\|$	$\left\|\frac{\Delta D_1}{D_1}\right\|$	$\left\|\frac{\Delta D_2}{D_2}\right\|$	$\left\|\frac{\Delta D_3}{D_3}\right\|$
9	0.135(−8)	0.496(−6)	0.267(8)	0.377(−9)	0.266(−8)	0.641(−8)	0.342(−8)
12	0.587(−8)	0.713(−7)	0.992(6)	0.337(−10)	0.239(−9)	0.578(−9)	0.305(−9)
16	0.772(−8)	0.189(−7)	0.679(6)	0.729(−11)	0.520(−10)	0.127(−9)	0.655(−10)
24	0.839(−8)	0.459(−8)	0.669(6)	0.169(−11)	0.121(−10)	0.296(−10)	0.152(−10)
32	0.849(−8)	0.462(−8)	0.669(6)	0.769(−11)	0.550(−11)	0.134(−10)	0.695(−11)

Table 6
The error norms and condition number by the BAM for $L = 34$ using the Gaussian rule with six nodes

N	$\|\varepsilon\|_B$	$\|\varepsilon\|_{\infty,\overline{AB}}$	Cond.(**A**)	$\left\|\frac{\Delta D_0}{D_0}\right\|$	$\left\|\frac{\Delta D_1}{D_1}\right\|$	$\left\|\frac{\Delta D_2}{D_2}\right\|$	$\left\|\frac{\Delta D_3}{D_3}\right\|$
12	0.359(−8)	0.721(−8)	0.675(6)	0.531(−13)	0.646(−12)	0.405(−11)	0.868(−11)
18	0.494(−8)	0.629(−8)	0.679(6)	0.468(−14)	0.211(−14)	0.620(−13)	0.352(−14)
24	0.491(−8)	0.530(−8)	0.679(6)	0.567(−15)	0.324(−15)	0.103(−14)	0.337(−13)
30	0.493(−8)	0.520(−8)	0.679(6)	0[a]	0.162(−15)	0.124(−14)	0.317(−13)
36	0.494(−8)	0.520(−8)	0.679(6)	0.850(−15)	0.324(−15)	0.103(−14)	0.308(−13)

[a] The error less than computer rounding errors in double precision.

Table 7
The error norms and condition number by BAM with different Gaussian rules as $L = 34$

Rule	N	$\|\varepsilon\|_B$	$\|\varepsilon\|_{\infty,\overline{AB}}$	Cond.(**A**)	$\left\|\frac{\Delta D_0}{D_0}\right\|$	$\left\|\frac{\Delta D_1}{D_1}\right\|$	$\left\|\frac{\Delta D_2}{D_2}\right\|$	$\left\|\frac{\Delta D_3}{D_3}\right\|$
Gauss(1)	24	0.839(−8)	0.459(−8)	0.606(6)	0.169(−11)	0.121(−10)	0.296(−10)	0.152(−10)
Gauss(2)	24	0.854(−8)	0.369(−8)	0.672(6)	0.708(−13)	0.512(−12)	0.133(−11)	0.106(−11)
Gauss(4)	24	0.610(−8)	0.540(−8)	0.679(6)	0.425(−15)	0.535(−14)	0.641(−13)	0.755(−13)
Gauss(6)	30	0.493(−8)	0.520(−8)	0.679(6)	0[a]	0.162(−15)	0.124(−14)	0.317(−13)
Gauss(8)	24	0.428(−8)	0.519(−8)	0.679(6)	0.142(−15)	0.648(−15)	0.618(−15)	0.315(−13)
Gauss(10)	20	0.639(−8)	0.521(−8)	0.679(6)	0.142(−15)	0[a]	0.412(−15)	0.308(−13)

[a] The error less than computer rounding errors in double precision.

the Gaussian rules of high order do not reduce the errors $\|\epsilon\|_B$, but do improve accuracy of leading coefficients. For $L = 34$ and $N = 30$, the exact leading coefficients $\widetilde{D}_0$ is obtained by the Gaussian rule of six nodes, see Table 8,

$$\widetilde{D}_0 = 401.162453745234416. \tag{6.32}$$

Compared (6.32) with the more accurate value [13] using higher precision by Mathematica, the relative error is less than the rounding error of double precision! This implies that $\widetilde{D}_0$ in (6.32) has 17 decimal significant digits.

6.4. Effective condition number and discussions

In real computation, the over-determined equations (6.31) and (2.33) are solved, which involve only the second and first order derivatives of the solutions respectively. Usually, the traditional condition number (2.35) for the spectral methods and the collocation methods is large, see Tables 1–7. To improve the number stability, we may invoke the preconditioning approaches, such as the preconditioned Krylov space CG methods. However, the real instability of the collocation methods for the given problems may not be so severe as the large Cond. displays. The new effective condition number is proposed to provide a better upper bound of condition number, which may be fairly small, to display a good stability.

The effective condition number was first studied in Chan and Foulser [7] and Christiansen and Hansen [8]. In fact, the definition of Cond. by (2.35) occurs only at the worst case, which may not happen for the real problems discussed. Let us take (6.31) as an example. Suppose that the matrix $\mathbf{F} \in R^{m\times n}$ $(m \geqslant n)$ is decomposed as [10],

$$\mathbf{F} = \mathbf{U\Sigma V}^{\mathrm{T}}, \tag{6.33}$$

where $\mathbf{U} \in R^{m\times m}$ and $\mathbf{V} \in R^{n\times n}$ are two orthogonal matrices, and $\mathbf{\Sigma} \in R^{m\times n}$ is the diagonal matrix with the declined singular values

$$\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n > 0. \tag{6.34}$$

Denote $\beta_i = \vec{\mathbf{u}}_{\mathbf{i}}^{\mathrm{T}}\vec{\mathbf{b}}$, where the orthogonal vectors $\vec{\mathbf{u}}_{\mathbf{i}}$ are given in $\mathbf{U} = (\vec{\mathbf{u}}_1, \ldots, \vec{\mathbf{u}}_{\mathbf{m}})$. Then the new effective condition number can be defined from Li et al. [15],

$$\mathrm{Cond\_eff} = \frac{1}{\sigma_n}\frac{\|\vec{\mathbf{b}}\|}{\sqrt{\left(\frac{\beta_1}{\sigma_1}\right)^2 + \cdots + \left(\frac{\beta_n}{\sigma_n}\right)^2}}, \tag{6.35}$$

where $\|\vec{\mathbf{b}}\|$ is the Euclidean norm, and $\|\vec{\mathbf{b}}\| = \sqrt{\sum_{i=1}^{m}\beta_i^2}$. Obviously, when there occurs the special case as $\beta_2 = \cdots = \beta_m = 0$, Eq. (6.35) leads to the traditional condition number (2.35). Note that the formula in (6.35) is somehow different from that given in [8], but it is easily computed. Moreover, when $\beta_n \neq 0$ we have from (6.35)

$$\mathrm{Cond\_eff} \leqslant \frac{\|\vec{\mathbf{b}}\|}{|\beta_n|}. \tag{6.36}$$

Table 8
The leading coefficients by the BAM with Gaussian rule of six nodes as $L = 34$ and $N = 30$

ℓ	$\widetilde{D}_\ell$
0	401.162453745234416
1	87.6559201950879299
2	17.2379150794467897
3	−8.0712152596987790
4	1.44027271702238968
5	0.331054885920006037
6	0.275437344507860671
7	−0.869329945041107943(−1)
8	0.336048784027428854(−1)
9	0.153843744594011413(−1)
10	0.730230164737157971(−2)
11	−0.318411361654662899(−2)
12	0.122064586154974736(−2)
13	0.530965295822850803(−3)
14	0.271512022889081647(−3)
15	−0.120045043773287966(−3)
16	0.505389241414919585(−4)
17	0.231662561135488172(−4)
18	0.115348467265589439(−4)
19	−0.529323807785491411(−5)
20	0.228975882995988624(−5)
21	0.106239406374917051(−5)
22	0.530725263258556923(−6)
23	−0.245074785537844696(−6)
24	0.108644983229739802(−6)
25	0.510347415146524412(−7)
26	0.254050384217598898(−7)
27	−0.110464929421918792(−7)
28	0.493426255784041972(−8)
29	0.232829745036186828(−8)
30	0.115208023942516515(−8)
31	−0.345561696019388690(−9)
32	0.153086899837533823(−9)
33	0.722770554189099639(−10)
34	0.352933005315648864(−10)

Table 9
The singular values σ_i and the coefficients β_i for the solution in Table 8 resulting from the BAM, where the Cond. = 0.679(6) and Cond_eff = 30.2

i	σ_i	β_i	i	σ_i	β_i
0	0.158(5)	0.420(2)	18	0.156(1)	−0.375(2)
1	0.121(5)	0.610(2)	19	0.126(1)	−0.602(1)
2	0.846(4)	0.133(2)	20	0.113(1)	−0.101(3)
3	0.595(4)	0.274(1)	21	0.974	0.130(2)
4	0.558(3)	0.584(2)	22	0.827	−0.117(3)
5	0.386(3)	0.246(2)	23	0.720	0.144(3)
6	0.269(3)	0.278(1)	24	0.677	−0.747(2)
7	0.195(3)	−0.177(2)	25	0.560	0.295(2)
8	0.513(2)	−0.591(2)	26	0.463	0.243(2)
9	0.345(2)	−0.544(1)	27	0.368	−0.180(2)
10	0.249(2)	0.146(2)	28	0.305	−0.136(2)
11	0.189(2)	−0.320(2)	29	0.249	0.134(2)
12	0.931(1)	−0.554(2)	30	0.188	−0.120(2)
13	0.619(1)	0.231(2)	31	0.141	−0.898(1)
14	0.470(1)	−0.421(2)	32	0.102	−0.908(1)
15	0.381(1)	−0.416(2)	33	0.556(−1)	0.815(1)
16	0.267(1)	−0.306(2)	34	0.233(−1)	−0.440(1)
17	0.202(1)	0.777(2)			

Hence we define the simplified effective condition number (see [15])

$$\text{Cond\_EE} = \frac{\|\vec{\mathbf{b}}\|}{|\beta_n|}, \quad \beta_n \neq 0. \tag{6.37}$$

More exploration on the effective condition number and its applications are given in Li et al. [15].

Let us evaluate the Cond_eff for the solution given in Table 8. The singular values σ_i of $\mathbf{F}$ and the coefficients β_i are listed in Table 9. Based on Table 9, we obtain the effective condition numbers, the traditional condition number, and their ratios,

$$\begin{aligned} &\text{Cond\_eff} = 30.2, \quad \text{Cond\_EE} = 65.7, \quad \text{Cond.} = 0.679(6), \\ &\frac{\text{Cond.}}{\text{Cond\_eff}} = 0.225(5), \quad \frac{\text{Cond.}}{\text{Cond\_EE}} = 0.103(5). \end{aligned} \tag{6.38}$$

The fact that the effective condition number is just 30.2 displays a good stability, and explains very well the high accuracy of the numerical solutions obtained and the leading coefficient $\widetilde{D}_0$ with 17 significant digits. In general, the leading coefficient D_0 in Table 7 may have 16 significant digits, and occasionally, D_0 has 17 significant digits due to the cancellation of rounding errors. The above arguments on effective number condition are also valid for the collocation methods used in Sections 6.1 and 6.2.

To close this section, let us first comment the assumptions **A2** and **A3**.

Remark 6.1. Usually, the solution of (2.1)–(2.3) is highly smooth inside of S, but less smooth on ∂S. In particular, for concave corners of polygons or the intersection points of the Dirichlet and Neumann boundary conditions, the solution near the boundary nodes is singular with infinite derivatives. In this case, some special treatments should be solicited. For Motz's problem with the singular origin in Section 6.3, the collocation equations are established at the nodes Q_i far from the singular origin, see (6.28)–(6.30). Hence assumptions **A2** in (2.5) can be relaxed to

$$v^{\pm} \in H^{1+\delta}(S) \cap C^2(D), \quad 0 < \delta < 1, \tag{6.39}$$

where the subdomain $D \subset S$ is far from the singularity. Moreover, when the singular functions or singular particular solution $v^{\pm}$ are chosen, assumption **A3** may also hold. Hence, the collocation methods may also be applied to singularity problems if suitable treatments are used.

7. Final remarks

1. In this paper, the collocation method is treated as the least squares method involving integration approximation. We employ the FEM theory to develop the theoretical analysis of CMs, in which the key analysis for the CMs is to prove the new V_h-elliptic inequality.
2. Three typical boundary conditions, Dirichlet, Neumann and Robin, can be handled well by the techniques of CMs in this paper. The number of collocation nodes may be chosen to be larger, even much larger than the number of radial

basis functions (i.e., source points). The collocation nodes are, indeed, the integration nodes of the rules used. Based on integration approximation, not only can the collocation nodes be easily located, but also the error analysis has been developed, see Sections 2 and 3.

3. Three computational examples in Section 6 show exponentially convergent rates: $\|u - v\|_{k,S} = O(\lambda^L)$, $k = 0, 1, 0 < \lambda < 1$, which verify perfectly the analysis made. Section 6.1 displays that the CM can be applied to many kinds of admissible functions, such as radial basis functions. The detailed analysis is given in Hu et al. [12].
4. Piecewise admissible functions can be used in the CM, both the analysis and the computation are provided in this paper, to enable the CM to be more flexible to complicated geometric domains for general PDEs, because different admissible functions can be chosen in different subdomains. Such an idea is also similar to the *p*-version of Babuška and Guo [2], and the analysis of the CM may also be extended to singularity problems by following [2].
5. Note that the real instability of the collocation methods for the given problems may not be so severe as the large Cond. displays. In Section 6, the new effective condition number is proposed to provide a better upper bound of condition number. The fairly small defective condition number for the real problems indicates a good stability of the collocation methods, and explains well the good computed results in our numerical experiments. Of course, some preconditioning techniques can also be used to improve the numerical stability.
6. In Remark 6.1, Assumptions **A2** and **A3** are discussed, which may be relaxed for singularity problems if some special treatments are used.
7. The BAM in [13,14] is a special case of the CM in this paper. The numerical results of Section 6.3 are better than those in [13,14]. The Gaussian rules with high orders are used to raise the accuracy of the leading coefficient D_0 obtained by the BAM. $\widetilde{D}_0$ in (6.32) is exact, in the sense of the errors less than the rounding errors of double precision, compared with the more accurate value of $\widetilde{D}_0$ in [13]. This new discovery will change the evaluation of the BAM given in [13], where, based on the numerical results in [14] using the centroid rule, "*BAM may produce the best global solutions*", and "*the conformal transformation method* (*CTM*) *is the most accurate method for leading coefficients*", see [13, p. 133]. Now we may conclude that for Motz's problem, the BAM (by the Gaussian rule with six nodes) is the most accurate method not only for the global solutions but also for the leading coefficient, $\widetilde{D}_0$.

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