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Superconvergence of solution derivatives of the Shortley–Weller difference approximation to Poisson's equation with singularities on polygonal domains

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This paper is dedicated to Professor Tetsure Yamamoto on the occasion of his 70th birthday.

Abstract

This paper presents a superconvergence analysis for the Shortley–Weller finite difference approximation of Poisson's equation with unbounded derivatives on a polygonal domain. In this analysis, we first formulate the method as a special finite element/volume method. We then analyze the convergence of the method in a finite element framework. An $O(h^{1.5})$-order superconvergence is derived for the solution derivatives in a discrete H^1 norm.

MSC: 65N10; 65N30

Keywords: Superconvergence; Solution derivatives; Boundary singularity; Finite difference method; Stretching function; Shortley–Weller approximation; Poisson's equation; Polygonal domains

1. Introduction

In this paper, we continue the study in [9,6] on superconvergence of solution derivatives of the Shortley–Weller finite approximations to the Poisson equation with the Dirichlet boundary condition

$$-\Delta u = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f(x, y), \quad (x, y) \in S, \tag{1.1}$$

$$u = g(x, y), \quad (x, y) \in \Gamma, \tag{1.2}$$

where $S \subset \mathbb{R}^2$ is a polygonal domain, Γ denotes the boundary of S and f and g are given functions. The superconvergence rates of orders $O(h^2)$ and $O(h^{1.5})$ of the solution derivatives in a discrete H^1 norm have been established

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for smooth problems on, respectively, rectangular and polygonal domains (cf. [9]). For problems with unbounded derivatives near Γ, the $O(h^2)$-order superconvergence of solution derivatives in the discrete energy norm has been achieved only for rectangular domains (cf. [6]). It is clear that in practice, rectangular domains are very restrictive, and polygonal domains are more desirable. In this paper we shall extend the analysis in [6] to problems on polygonal domains. In this investigation, two challenging problems become apparent: (1) how to find suitable local refinements near the boundary using triangles and rectangles, and (2) how to estimate error bounds resulting from rectangular and triangular elements in a suitable partition.

For the first question, we adopt the ideas of combination in [5] to split a polygonal domain, S, into several non-overlapped subpolygons. Each sub-polygon can be partitioned into a mesh by a system of difference grids. To make the global partition consistent, the mesh points along an interior boundary segment need to be common to the partitions of the two neighboring subpolygons. Since the piecewise bilinear and linear admissible functions are continuous on the entire domain, no errors result from the interior boundary segments, i.e., the method is conforming. Obviously, the combined difference grids used make the finite difference method (FDM) more flexible.

As to the second problem, because no errors occur from the divergent integrals on triangular elements involving the unbounded derivatives, we simply need to estimate the errors from the divergent integrals on rectangular elements. New analysis is needed only to estimate errors from the approximate integrals on triangular elements involving the non-homogeneous term of the equation. By following the arguments in [6], an $O(h^{1.5})$ superconvergence can be achieved for the Shortley–Weller difference approximation.

This rest of paper is organized as follows. In the next section, we will first present the Shortley–Weller difference scheme for the Poisson equation. We will then formulate the method as a special finite element method (FEM) and a finite volume method (FVM). In Section 3, an error analysis for the numerical method is presented and the superconvergence of order $O(h^{1.5})$ in a discrete norm is obtained. In Sections 4 and 5, we estimate the errors due to the non-homogeneous term of equation on triangular elements and the those of the divergent integrals on rectangular elements respectively.

Before further discussion, we first formulate (1.1) and (1.2) as a variational problem.

Let $L^2(S)$ be the space of square integrable functions on S, and let $H^1(S)$ be the usual Sobolev space. We put $H_0^1 := \{v\colon\ v \in H^1(S),\ v|_\Gamma = 0\}$. The variational problem corresponding to (1.1)–(1.2) can be expressed by: Find $u \in H^1(S)$ such that

$$a_h(u, v) = f_h(v), \quad \forall v \in H_0^1(S),$$

where the bilinear and linear forms are defined respectively by

$$a_h(u, v) = \iint_S \nabla u \nabla v \, ds, \qquad f_h(v) = \iint_S f v \, ds.$$

2. The finite difference methods

In this section we will progressively construct the Shortley–Weller scheme [1,13] on various difference meshes. We will start with the basis Shortley–Weller scheme.

2.1. The basis Shortley–Weller difference approximation

To construct the finite difference scheme, we first define a finite difference mesh for the solution domain S and its boundary Γ as follows. Since S is polygonal, we assume that it is divided into a mesh containing rectangles and triangles with the mesh lines either parallel to one of the axes or on Γ. In this mesh, all the triangles have at least one side on Γ. Let I be the (double) index set of this mesh and $X = (x_i, y_j)$, $\forall (i, j) \in I$ be the set of mesh nodes on $\overline{S}$. We now split the nodal set X into two disjoint subsets: the set containing nodes in S, denoted by S_h, and that on Γ, denoted by Γ_h. The index subsets of I corresponding to S_h and Γ_h are denoted by I_S and I_Γ, respectively. For any feasible indices i and j, let $h_i = x_{i+1} - x_i$ and $k_j = y_{j+1} - y_j$ be the step sizes along the two directions, respectively. We put $h = \max_{i,j}\{h_i, k_j\}$. In what follows we use $\square_{ij}$ and $\triangle_{ij}$ to denote respectively the rectangular and triangular element associated with (i, j) as shown in Fig. 1. For these elements, we have

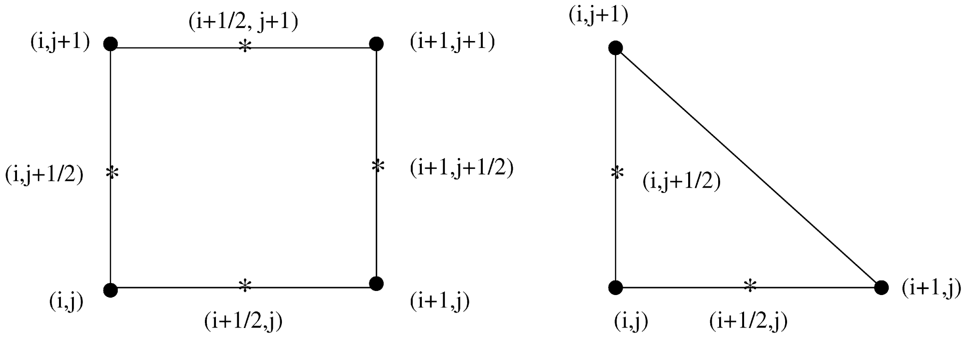

Fig. 1. A rectangle $\square_{ij}$ and a $\triangle_{ij}$.

$$S = S_\square \cup S_\triangle := \Big(\bigcup_{ij} \square_{ij}\Big) \cup \Big(\bigcup_{ij} \triangle_{ij}\Big). \tag{2.1}$$

As constructed, all elements in $S_\triangle$ are right-angled triangles and located near the boundary Γ. Therefore, the two nodes other than (x_i, y_j) of $\triangle_{ij} \in S_\triangle$ are in Γ_h. Obviously, the total number of triangles is much less than the number of rectangles in this mesh. As shown in [5], the conventional finite difference method can be formulated as a special finite element method using piecewise bilinear and linear interpolating functions, $v(x, y)$, on $\square_{ij}$ and $\triangle_{ij}$ defined respectively by,

$$\begin{aligned} v(x,y) = \frac{1}{h_i k_j}\big\{&(x_{i+1}-x)(y_{j+1}-y)v_{i,j} + (x-x_i)(y_{j+1}-y)v_{i+1,j} \\ &+ (x_{i+1}-x)(y-y_j)v_{i,j+1} + (x-x_i)(y-y_j)v_{i+1,j+1}\big\}, \quad (x,y)\in \square_{ij}, \end{aligned} \tag{2.2}$$

and

$$v(x,y) = v_{i,j} + \frac{(x-x_i)}{h_i}(v_{i+1,j} - v_{i,j}) + \frac{(y-y_j)}{k_j}(v_{i,j+1} - v_{i,j}), \quad (x,y) \in \triangle_{ij}, \tag{2.3}$$

where $v_{k,\ell}$ denotes the nodal value of v at (x_k, y_ℓ). Let $V_h \subseteq H^1(S)$ denote a finite dimensional space of the piecewise bilinear and linear functions v of (2.2) and (2.3) satisfying (1.2), and we denote by V_h^0 the subset of V_h satisfying $v = 0$ on Γ. The FDM with the quadrature approximations to the line and area integrals are defined by: Find $u_h \in V_h$ such that

$$\hat{a}_h(u_h, v) = \hat{f}_h(v), \quad \forall v \in V_h^0, \tag{2.4}$$

where

$$\hat{a}_h(u,v) = \widehat{\iint_S} \nabla u \nabla v\, ds = \sum_{ij\in I_S}\Big[\widehat{\iint_{\square_{ij}}} \nabla u \nabla v\, ds + \widehat{\iint_{\triangle_{ij}}} \nabla u \nabla v\, ds\Big], \tag{2.5}$$

$$\hat{f}_h(v) = \widehat{\iint_S} f v\, ds = \sum_{ij\in I_S}\Big[\widehat{\iint_{\square_{ij}}} f v\, ds + \widehat{\iint_{\triangle_{ij}}} f v\, ds\Big], \tag{2.6}$$

where $ij \in I_S$ means the grids $(x_i, y_j) = (i, j) \in S_h$. The approximate integrals in (2.5) and (2.6) over rectangles $\square_{ij}$ are evaluated by the following quadrature rules

$$\widehat{\iint_{\square_{ij}}} \nabla u \nabla v\, ds = \widehat{\iint_{\square_{ij}}} u_x v_x\, ds + \widehat{\iint_{\square_{ij}}} u_y v_y\, ds, \tag{2.7}$$

$$\widehat{\iint_{\square_{ij}}} u_x v_x\, ds = \frac{h_i k_j}{2}\Big[u_x\Big(i+\frac12, j\Big)v_x\Big(i+\frac12, j\Big) + u_x\Big(i+\frac12, j+1\Big)v_x\Big(i+\frac12, j+1\Big)\Big], \tag{2.8}$$

$$\widehat{\iint_{\square_{ij}}} u_y v_y \, ds = \frac{h_i k_j}{2}\left[u_y\left(i, j+\frac{1}{2}\right)v_y\left(i, j+\frac{1}{2}\right) + u_y\left(i+1, j+\frac{1}{2}\right)v_y\left(i+1, j+\frac{1}{2}\right)\right], \tag{2.9}$$

$$\widehat{\iint_{\square_{ij}}} f v \, ds = \frac{h_i k_j}{4}\left[f_{ij}v_{ij} + f_{i+1,j}v_{i+1,j} + f_{i,j+1}v_{i,j+1} + f_{i+1,j+1}v_{i+1,j+1}\right], \tag{2.10}$$

where $w_{ij} = w(x_i, y_j)$ for any w and ij and

$$u_x\left(i+\frac{1}{2}, j\right) = \frac{u_{i+1,j} - u_{ij}}{h_i}, \qquad u_y\left(i, j+\frac{1}{2}\right) = \frac{u_{i,j+1} - u_{ij}}{k_j}$$

with the mesh nodes being defined in Fig. 1. Similarly, the integrals over a triangle $\triangle_{ij}$ used in (2.5) and (2.6) are approximated by

$$\widehat{\iint_{\triangle_{ij}}} \nabla u \nabla v \, ds = \frac{h_i k_j}{2}\left[u_x\left(i+\frac{1}{2}, j\right)v_x\left(i+\frac{1}{2}, j\right) + u_y\left(i, j+\frac{1}{2}\right)v_y\left(i, j+\frac{1}{2}\right)\right], \tag{2.11}$$

$$\widehat{\iint_{\triangle_{ij}}} f v \, ds = \frac{h_i k_j}{8}\left[2f_{ij}v_{ij} + f_{i+1,j}v_{i+1,j} + f_{i,j+1}v_{i,j+1}\right]. \tag{2.12}$$

From (2.4) and (2.7)–(2.12), the traditional Shortley–Weller difference scheme centered at the grid $(i, j) \in S_h$ is given by

$$-\frac{(k_{j-1}+k_j)}{2h_i}(u_{i+1,j} - u_{i,j}) - \frac{(k_{j-1}+k_j)}{2h_{i-1}}(u_{i-1,j} - u_{i,j}) - \frac{(h_{i-1}+h_i)}{2k_j}(u_{i,j+1} - u_{i,j})$$
$$- \frac{(h_{i-1}+h_i)}{2k_{j-1}}(u_{i,j-1} - u_{i,j}) = \frac{(h_{i-1}+h_i)(k_{j-1}+k_j)}{4} f_{i,j}.$$

Dividing both sides of the above equation by $\frac{(h_{i-1}+h_i)(k_{j-1}+k_j)}{4}$ gives the Shortley–Weller approximation to the equation.

For clarity, we will, in this paper, concentrate on the basic feature of the FDM in which the derivatives u_x and u_y are replaced approximately and straightforwardly by the divided differences. Hence, basic elements in the FDM must be rectangles $\square_{ij}$ and right angled triangles $\triangle_{ij}$. More general partitions of S into rectangles $\square_{ij}$ and triangles $\triangle_{ij}$ are also possible if we use the ideas of combined methods in [5]: Let S be divided by an interior boundary Γ_0 into several non-overlapped subdomains S_i, $i = 1, 2, \ldots, N$. Each S_i is further partitioned into rectangles and triangles. We assume that the grid points (i, j) on Γ_0 are common to the meshes on both sides of Γ_0. In this case our method is conforming because there is no discontinuity across Γ_0 in the piecewise linear and bilinear interpolating functions.

2.2. Reformulation of the FDM as a FVM

From the previous subsection we see that the FDM can be regarded as a special FEM. This formulation has the following three characteristics:

(1) Based on the partition (2.1), we see that the right-angled triangles in the mesh are located only near exterior boundary Γ and interior boundary Γ_0. Hence, the partition contains predominantly rectangles.
(2) In the FEM formulation, we use bilinear and linear basis functions corresponding to $\square_{ij}$ and $\triangle_{ij}$ respectively, and thus it is a combination of bilinear and linear elements.
(3) Special quadrature rules approximating the integrals are used to facilitate formulation of difference equations. In this formulation, the derivatives are approximated by the divided differences.

As will be demonstrated later in this subsection, it is also possible to reformulate the FDM as a finite volume method which enables us to write down the difference equations straightforwardly. Merits and drawbacks are twins. Compared with FEMs, the merit of the FDM is the facile formulation of linear algebraic equations, but the drawback is it is less flexible than the linear FEM in which rather arbitrary $\triangle_{ij}$ can be chosen.

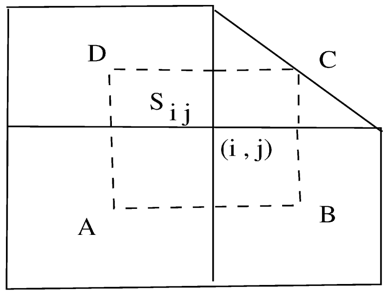

Fig. 2. A control regions S_{ij}.

Let us consider two kinds of FEMs: find u_h and $u_h^E \in V_h$ such that

$$\widehat{\iint_S} \nabla u_h \nabla v = \widehat{\iint_S} f v, \quad \forall v \in V_h^0, \tag{2.13}$$

$$\iint_S \nabla u_h^E \nabla v = \widehat{\iint_S} f v, \quad \forall v \in V_h^0, \tag{2.14}$$

where u_h and u_h^E are the solutions from the Shortley–Weller approximation and the linear FEM, respectively. Rule (2.12) for $\triangle_{ij}$ is used in the Shortley–Weller approximation. the domain S may be partitioned into a mesh containing right angled triangles $\triangle_{ij}$ only, i.e., $S = S_\triangle$, which can be done by splitting each $\square_{ij} \in S_\square$ into two triangles $\triangle_{ij}^+$ and $\triangle_{ij}^-$ by a diagonal of $\square_{ij}$. Note that the integral $\iint_S \nabla u_h^E \nabla v$, can be evaluated exactly because both u_h^E and v are the linear functions. However, $\iint_{\triangle_{ij}} f v$ is usually evaluated approximately by some quadrature rules because f may be arbitrary. Let us take the triangle $\triangle_{ij}$ as an example. A possible quadrature rule is

$$\widehat{\iint_{\triangle_{ij}}} f v \, ds = \frac{h_i k_j}{6} [f_{ij} v_{ij} + f_{i+1,j} v_{i+1,j} + f_{i,j+1} v_{i,j+1}].$$

Note that for the linear FEM, even $\triangle_{ij}$'s are not right angled triangles, the integral $\iint_S \nabla u_h^E \nabla v$, can still be evaluated exactly (cf., for example, [2,10]).

It is easy to show that the linear systems associated with (2.13)–(2.14), are respectively

$$\mathbf{A}\vec{x} = \vec{b}_1, \quad \text{and} \quad \mathbf{A}\vec{x}^E = \vec{b}_2,$$

where $\mathbf{A}$ is the system matrix arising from both (2.13) and (2.14) and the unknown vectors $\vec{x}$ and $\vec{x}^E$ consist of the nodal values of u_h and u_h^E, respectively. The above explanation links the FDM to the FEM.

Next, we make a link between the FDM and a special FVM. In [8], the finite volume method can also be viewed as a special FEM based on a Delaunay triangulation. To formulate the FDM as a finite volume scheme, we define a partition of domain S dual to the original by the bisectors of the edges of each $\square_{ij}$ and $\triangle_{ij}$. We denote this dual partition by

$$S = S_C := \left(\bigcup_{ij} S_{ij} \right), \tag{2.15}$$

where each S_{ij} is called the control region of (x_i, y_j), see Fig. 2. Integrating the left-hand side of (1.1) by parts on S_{ij}, we have

$$-\int_{\partial S_{ij}} \frac{\partial u}{\partial n} = \iint_{S_{ij}} f,$$

where n denotes the unit vector outward normal to ∂S_{ij} and ∂S_{ij} denotes the boundary of S_{ij}. Applying the one-point quadrature rule and a proper quadrature rule (to be defined later) to the right-hand and left-hand sides of the above equation, respectively, gives

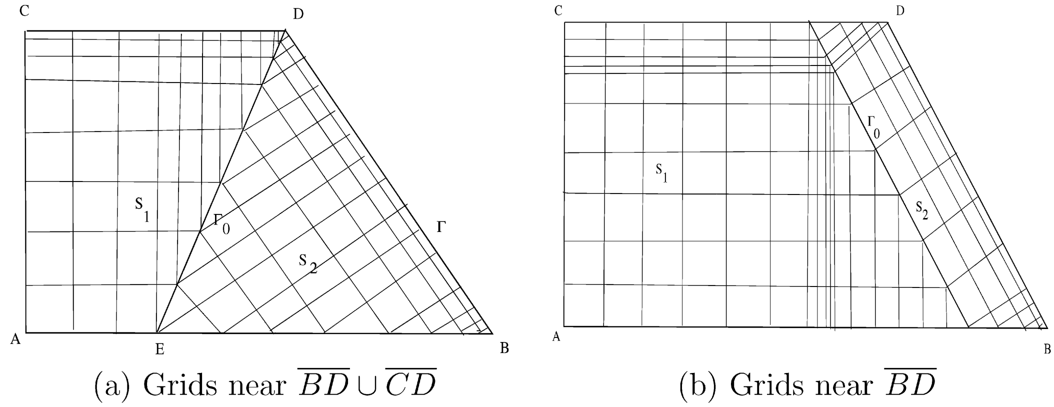

Fig. 3. Typical local refinements of difference grids.

$$-\widehat{\int_{\partial S_{ij}}} \frac{\partial u}{\partial n} = |S_{ij}| f_{ij}, \tag{2.16}$$

where $|S_{ij}|$ denotes the area of S_{ij}. To define the approximation on the left-hand side of (2.16), we take tile in Fig. 2 as an example. In this case, we have $|S_{ij}| = \frac{1}{4}(h_{i-1}+h_i)(k_{j-1}+k_j)$. Since ∂S_{ij} consists of four segments, the right-hand side of (2.16) can be approximated by the following finite difference scheme

$$\widehat{\int_{\partial S_{ij}}} \frac{\partial u}{\partial n} = \widehat{\int_{\overline{AB}}} \frac{\partial u}{\partial n} + \widehat{\int_{\overline{BC}}} \frac{\partial u}{\partial n} + \widehat{\int_{\overline{CD}}} \frac{\partial u}{\partial n} + \widehat{\int_{\overline{DA}}} \frac{\partial u}{\partial n},$$

where

$$\widehat{\int_{\overline{AB}}} \frac{\partial u}{\partial n} = \frac{u_{i,j-1} - u_{ij}}{k_{j-1}} \left(\frac{h_{i-1}+h_i}{2}\right), \quad \widehat{\int_{\overline{BC}}} \frac{\partial u}{\partial n} = \frac{u_{i+1,j} - u_{ij}}{h_i} \left(\frac{k_{j-1}+k_j}{2}\right),$$

$$\widehat{\int_{\overline{CD}}} \frac{\partial u}{\partial n} = \frac{u_{i,j+1} - u_{ij}}{k_j} \left(\frac{h_{i-1}+h_i}{2}\right), \quad \widehat{\int_{\overline{DA}}} \frac{\partial u}{\partial n} = \frac{u_{i-1,j} - u_{ij}}{h_{i-1}} \left(\frac{k_{j-1}+k_j}{2}\right).$$

Clearly, Eq. (2.16) with the above quadrature rules becomes the Shortley–Weller difference approximation.

2.3. Difference scheme for problems with singularities

In this subsection we will modify the previous numerical method for (1.1) with unbounded derivatives on $\Gamma_U \subseteq \Gamma$. This includes a meshing technique and a modification of the difference scheme, as given below.

To mesh S, we follow the following rules.

- We divide S into several subdomains by an interior boundary Γ_0, and mesh each of the subdomain separately.
- For the subdomain containing part of Γ_U, we mesh it mesh using mesh lines parallel or perpendicular to that part of Γ_U. The mesh lines parallel to Γ_U are normally graded according to a rule to be defined.
- Make sure that all the mesh nodes on Γ_0 are common to the meshes on both sides of Γ_0 to avoid the case of non-conformity.

Let us demonstrate this using the subdomain in Fig. 3(a), in which the solution derivatives are assumed to be unbounded on two edges, $\overline{CD}$ and $\overline{DB}$. For this case, we split this subdomain into two subsubdomains, denoted S_1 and S_2, by $\Gamma_0 = \overline{DE}$ that is chosen such that $\angle CDE \approx \angle EDB$. Then two local Cartesian coordinate systems are used in S_1 and S_2 respectively, and the local non-uniform meshes are used along $\overline{CD}$ and $\overline{BD}$. Fig. 3(b) also illustrates a

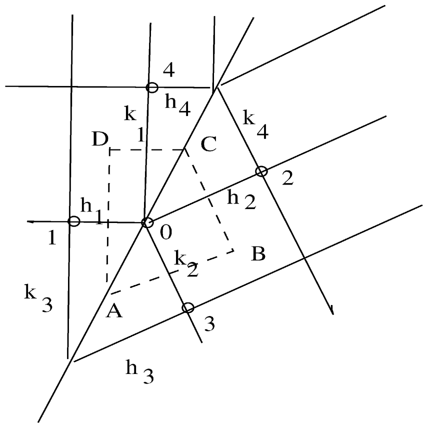

Fig. 4. The control region S_{ij}.

Table 1
Comparisons of FDM with FVM and FEM

Methods	FDM	FVM	FEM
Basic elements	Rectangles and Right triangles	Delaunay triangles	Triangles
Formation of difference equations	Simplest	Modest	Complicated
Applications	Limited	Modest	Wide

partition used for a problem which has unbounded derivative along the edge $\overline{BD}$. Clearly, a mesh constructed in this way contains mostly rectangular elements. Triangular elements are needed only near Γ and Γ_0.

Since the above meshing technique normally creates pairs of triangular element sharing a common edge on Γ_0 the numerical scheme in Sections 2.1 or 2.2 needs to be modified. Let us take a typical case shown in Fig. 4 for demonstration which contains two rectangular and four triangular elements. Based on the FVM interpretation (2.16) and using the local index notation, we construct the difference scheme associated with Node 0 as follows

$$\frac{1}{2}\left\{\frac{k_1+k_3}{h_1}(v_0-v_1)+\frac{k_2+k_4}{h_2}(v_0-v_2)+\frac{h_2+h_3}{k_2}(v_0-v_3)+\frac{h_1+h_4}{k_1}(v_0-v_4)\right\}$$
$$=\frac{1}{8}\left\{(h_1+h_4)(k_1+k_3)+(h_2+h_3)(k_2+k_4)\right\}f_0,$$

where v_0 denotes the v_{ij} and v_k, $k=1,2,3,4$, are the nodal values of v at the neighboring grids. Difference schemes for other cases can be constructed similarly to the above.

A linkage is explored for three popular methods, the FDM, the FDM and the FVM, as given in Table 1. For different problems or different requirements, different numerical methods may be chosen [5]. As indicated in Table 1, the FDM has a limitation of grid partition for arbitrary domains because it is based on rectangles and right triangles. If such characteristics can not be retained, such a method is not of FDM. For the corner singularities, the FEM is obviously superior to the FDM. However, for line singularities, the FDM outperforms the FEM due to its simplicity, and due to the error analysis given in this paper. Each method has its own merits and drawbacks. The purpose of this research is to explore the merits, particularly the superconvergence phenomenon, of the FDM for PDEs with line singularities.

3. Error analysis of the method

In this section we present the main results of the paper on the superconvergence of the solution of the finite difference method in discussed Section 2. This error analysis is carried out using the following discrete H^1 norm:

$$\overline{\|v\|}_1^2 := \overline{\|v\|}_{1,S}^2 = \overline{|v|}_{1,S}^2 + \overline{\|v\|}_{0,S}^2, \quad \forall v \in H^1(S)$$

where $\overline{|v|}_{1,S}$ and $\overline{\|v\|}_{0,S}$ are defined respectively by

$$\overline{|v|}_1^2 = \overline{|v|}_{1,S}^2 = \sum_{ij \in I_S} \left[\widehat{\iint_{\square_{ij}}} (\nabla v)^2 \, ds + \widehat{\iint_{\triangle_{ij}}} (\nabla v)^2 \, ds \right],$$

$$\overline{\|v\|}_0^2 = \overline{\|v\|}_{0,S}^2 = \sum_{ij \in I_S} \left[\widehat{\iint_{\square_{ij}}} v^2 \, ds + \widehat{\iint_{\triangle_{ij}}} v^2 \, ds \right].$$

Here the quadrature rules, $\widehat{\iint}_{\square_{ij}} (\nabla v)^2 \, ds$, $\widehat{\iint}_{\triangle_{ij}} (\nabla v)^2 \, ds$, $\widehat{\iint}_{\square_{ij}} v^2 \, ds$ and $\widehat{\iint}_{\triangle_{ij}} v^2 \, ds$ are given in Section 2.1.

It has been shown in [2], p. 196, that

$$\overline{\|u - u_h\|}_1 \leqslant \mathrm{C} \left\{ \overline{\|u - u_I\|}_1 + \sup_{w \in V_h^0} \frac{|(\iint_S - \widehat{\iint_S}) \nabla u_I \nabla w \, ds|}{\overline{\|w\|}_1} + \sup_{w \in V_h^0} \frac{|(\iint_S - \widehat{\iint_S}) f w \, ds|}{\overline{\|w\|}_1} \right\}, \tag{3.1}$$

where u_I is the V_h-interpolant of the true solution u. Since both u_I and w are linear on $\triangle_{ij}$, it is easy to verify that

$$\iint_{\triangle_{ij}} \nabla u_I \nabla w \, ds - \widehat{\iint_{\triangle_{ij}}} \nabla u_I \nabla w \, ds = 0.$$

Splitting each integral on the right-hand side of (3.1) into one over the union of rectangular elements and another over the union of the triangular elements and using the above equality, we have from (3.1)

$$\begin{aligned} \overline{\|u - u_h\|}_1 \leqslant{}& C \left\{ \sup_{w \in V_h^0} \frac{|(\iint_{S_\square} - \widehat{\iint_{S_\square}}) \nabla u_I \nabla w \, ds|}{\overline{\|w\|}_1} \right. \\ & \left. + \left(\overline{\|u - u_I\|}_1 + \sup_{w \in V_h^0} \frac{|(\iint_{S_\square} - \widehat{\iint_{S_\square}}) f w \, ds|}{\overline{\|w\|}_1} \right) + \sup_{w \in V_h^0} \frac{|(\iint_{S_\triangle} - \widehat{\iint_{S_\triangle}}) f w \, ds|}{\overline{\|w\|}_1} \right\} \\ =:{}& T + T_1 + T_2. \end{aligned} \tag{3.2}$$

The following assumptions characterize the natures of the singularity in the solution and the right-hand side of the continuous Poisson equation and the finite difference mesh near the boundary $\Gamma_U \subseteq \Gamma$.

A1: The derivatives of the solution u close to Γ_U satisfies (cf. [12,3]):

$$\sup_{0 < d(x,y) \ll 1} d^{i-\sigma}(x, y) \left| \frac{\partial^i}{\partial n^i} u(x, y) \right| \leqslant C_1, \quad i = 1, 2, 3, 4,$$

for some constants $\sigma > 0$ and $C_1 > 0$, independent of u, where d denotes the distance between $(x, y) \in S$ and Γ_U and n denotes the unit vector in the direction from (x, y) to the point on Γ_U closest to (x, y). Moreover, we assume that for $0 < r \ll 1$, u satisfies

$$\sup_{dist(P,Q) \leqslant r} \left| u(P) - u(Q) \right| \leqslant C r^\sigma,$$

where P and Q are two arbitrary points in S near Γ_U. In this paper, we assume

$$\sigma = \frac{1}{2} + \mu \tag{3.3}$$

for some $\mu > 0$. This is necessary for the derivative estimates used in the analysis.

A2: The function f satisfies the following properties (cf. [12,3]):

$$\sup_{0 < d \ll 1} d^{i-\nu}(x, y) \left| \frac{\partial^i}{\partial n^i} f(x, y) \right| \leqslant C_2, \quad i = 0, 1, 2,$$

$$\sup_{dist(P,Q) \leqslant r} \left| f(P) - f(Q) \right| \leqslant C r^\nu, \tag{3.4}$$

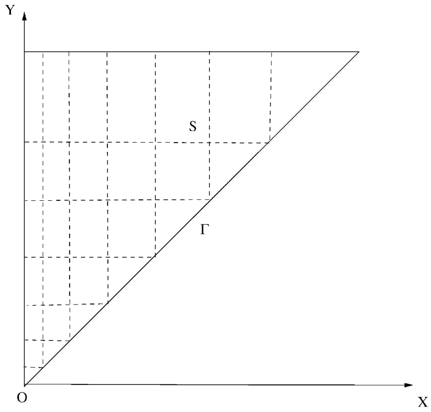

Fig. 5. Partition of x_i and y_j.

for some constants $\nu > 0$, $C_2 > 0$ and $0 < r \ll 1$. To make ν compatible with (3.3), we assume it satisfies

$$\nu = -\frac{3}{2} + \mu, \quad \mu > 0.$$

A3: We assume that the difference mesh near Γ_U is constructed locally so that the mesh lines are either parallel or perpendicular to one of the boundary segments in Γ_U. Furthermore, we assume that the mesh lines parallel to one segment of Γ_U are constructed non-uniformly according to the distances from the segment $d_\ell = \psi(\ell h)$, $\ell = 0, 1, \ldots, n$, for a positive integer n, where $h = \frac{1}{n}$ and $\psi(\cdot)$ is the stretching function on [0, 1] defined by

$$\psi(s) = \frac{1}{\int_0^1 z^p(1-z)^p\,dz} \int_0^s z^p(1-z)^p\,dz, \quad s \in [0, 1],$$

with $p > 0$ a (proper) constant (cf. [12,3]), whose choice will be given in the following theorems.

A triangulation family with the mesh parameter h, is said to be regular if for each triangle $\triangle_{ij}$ of the mesh we have if

$$\frac{\max\{l_{ij}^{(1)}, l_{ij}^{(2)}, l_{ij}^{(3)}\}}{\min\{l_{ij}^{(1)}, l_{ij}^{(2)}, l_{ij}^{(3)}\}} \leqslant C$$

for all $0 < h \leqslant h_0$, where $l_{ij}^{(k)}$, $k = 1, 2, 3$, denote the lengths of the three sides of $\triangle_{ij}$, h_0 is a positive constant and C is a positive constant, independent of the mesh sizes. The regularity implies that the interior angles of any $\triangle_{ij}$ are bounded below by a positive constant when $h \to 0^+$. However, for the meshes defined in the previous section some triangles having at least one side on Γ and Γ_0 may satisfy the above inequality because of the stretching function given in **A3**. Hence, we make the following assumption.

A4: The family of triangles located on Γ or Γ_0, using the local refinements techniques in **A3** is regular.

To make our proof notationally simple, we consider a special case of the above problem which has triangular solution domains of the shape depicted in Fig. 5. We simplify Assumptions **A1** and **A2** as the following two.

A5: Suppose that the solution u is sufficiently smooth except[1] for the derivatives with respect to x at $x = 0$, which are unbounded and satisfy

[1] This implies that up to and including 4th order derivatives with respect to y used are bounded. However, the derivatives with respect to x are constrained by (3.5).

$$\sup_{x\in(0,1)} x^{i-\sigma}\left|\frac{\partial^i}{\partial x^i}u(x,y)\right| \leqslant C, \quad i=1,2,3,4,$$
$$\sup_{x\in(0,1)} x^{i-\nu}\left|\frac{\partial^i}{\partial x^i}f(x,y)\right| \leqslant C, \quad i=0,1,2, \tag{3.5}$$

where $\sigma = \frac{1}{2}+\mu$, $\nu = -\frac{3}{2}+\mu$ and $0<\mu<\frac{3}{2}$.

Note that Assumptions **A1**, **A2** and **A5** are commonly used for problems with unbounded derivatives near the boundary, one popular type of boundary singularities (cf., for example, [3,4,14,6,11,12]). For this case, the application of the meshing technique in **A3** results in a mesh depicted in Fig. 5. This is given in the following assumption.

A6: Let both x_i and y_j of partition in S be chosen as in **A3**, see Fig. 5.

Finally, we have the following theorem, whose proof is deferred in Sections 4 and 5.

Theorem 3.1. *Let $S = S_\Box \cup S_\triangle$ and* **A1–A6** *be fulfilled. For $\sigma = \frac{1}{2}+\mu$ and $\nu = -\frac{3}{2}+\mu$, $\mu>0$, there exists the error bound of the solution u_h from the Shortley–Weller difference approximation*

$$\overline{\|u-u_h\|}_1 \leqslant T + T_1 + T_2 \leqslant Ch^r, \quad r \leqslant 1.5,$$

where $r = (p+1)\mu$ and T, T_1 and T_2 are defined in (3.2).

The equation $p+1 = \frac{1.5}{\mu}$ from Theorem 3.1 may not reachable, since the condition number $\mathrm{Cond} = \mathrm{O}(h_{\min}^{-2}) = \mathrm{O}(h^{-2(p+1)})$. From the error analysis, the solution singularity $u = \mathrm{O}(r^{\frac{1}{2}+\mu})$ with $\mu>0$ is allowable to achieve the optimal convergence rate $\mathrm{O}(h^{1.5})$. However, for stability, μ should be larger than μ_0 (>0) by noting that $\mu = 0.15$ gives $p+1 = 10$, leading to the huge condition number $\mathrm{Cond} = \mathrm{O}(h^{-20})$.

4. Bounds for T_1 and T_2

In this section, we will estimate bounds on T_1 and T_2 in (3.2). Bounds on T will be established in the next section. The following three lemmas have been established in [9,6].

Lemma 4.1. *Let u be the exact solution to* (1.1)–(1.2)*, and let u_I be the V_h-interpolant of u. Then, we have*

$$\overline{\|u-u_I\|}_1 = \left\{\sum_{ij\in I_S}\overline{\|u-u_I\|}^2_{1,\Box_{ij}} + \sum_{ij\in I_S}\overline{\|u-u_I\|}^2_{1,\triangle_{ij}}\right\}^{\frac{1}{2}},$$

and

$$\overline{|u-u_I|}^2_{1,\Box_{ij}} \leqslant \frac{1}{2}\iint_{\Box_{ij}} \kappa^2_{1,i}(x)\big(u^2_{xxx}(x,y_j)+u^2_{xxx}(x,y_{j+1})\big) + \kappa^2_{2,j}(y)\big(u^2_{yyy}(x_i,y)+u^2_{yyy}(x_{i+1},y)\big),$$

$$\overline{|u-u_I|}^2_{1,\triangle_{ij}} \leqslant \iint_{\triangle_{ij}} \kappa^2_{1,i}(x)u^2_{xxx}(x,y_j) + \kappa^2_{2,j}(y)u^2_{yyy}(x_i,y),$$

where $u_{x^iy^j} = \frac{\partial^{i+j}u}{\partial x^i\partial y^j}$, and

$$\kappa_{1,i}(x) = \begin{cases} \frac{1}{2}(x-x_i)^2 & \text{for } x_i \leqslant x \leqslant x_i+\frac{h_i}{2},\\ \frac{1}{2}(x_{i+1}-x)^2 & \text{for } x_i+\frac{h_i}{2} \leqslant x \leqslant x_{i+1},\end{cases}$$

$$\kappa_{2,j}(y) = \begin{cases} \frac{1}{2}(y-y_j)^2 & \text{for } y_j \leqslant y \leqslant y_j+\frac{k_j}{2},\\ \frac{1}{2}(y_{j+1}-y)^2 & \text{for } y_j+\frac{k_j}{2} \leqslant y \leqslant y_{j+1}.\end{cases}$$

Lemma 4.2. *There exists the bound*:

$$\left|\left(\iint_{S_\square} - \widehat{\iint_{S_\square}}\right) f w\right| \leqslant C\overline{|||f|||}_\square \times \overline{\|w\|}_{1,S_\square}, \quad w \in V_h^0,$$

where

$$\overline{|||f|||}_\square = \sqrt{\sum_{\square_{ij}\in S_\square} \widehat{\iint_{\square_{ij}}} p_{ij}^2 + \|q_{ij}\|^2_{-1,\square_{ij}}},$$

$$p_{ij}^2 = \left(f_x^2 + f_x^2(x, y_j) + f_x^2(x, y_{j+1})\right)(x-x_i)^2(x-x_{i+1})^2 + \left(f_y^2 + f_y^2(x_i, y) + f_y^2(x_{i+1}, y)\right)(y-y_j)^2(y-y_{j+1})^2,$$

$$q_{ij}^2 = \left(f_{xx}^2 + f_{xx}^2(x, y_j) + f_{xx}^2(x, y_{j+1})\right)(x-x_i)^2(x-x_{i+1})^2 + \left(f_{yy}^2 + f_{yy}^2(x_i, y) + f_{yy}^2(x_{i+1}, y)\right)(y-y_j)^2(y-y_{j+1})^2,$$

and

$$\|q_{ij}\|_{-1,\square_{ij}} \leqslant \sup_{w\in V_h^0} \frac{\iint_{\square_{ij}} q_{ij} w}{\|w\|_{1,\square_{ij}}}.$$

Combining Lemmas 4.1 and 4.2, we have following lemma.

Lemma 4.3. *Let* **A1**–**A3** *be given. There exists the error bound of* T_1,

$$T_1 \leqslant C\sum_{\ell=1}^{n}\left\{\ell^{2\mu(p+1)-5} \times h^{2\mu(p+1)}\right\}^{\frac{1}{2}},$$

where $h = \frac{1}{N}$, *as given in* **A3**. *Moreover, for* $\sigma = \frac{1}{2} + \mu$ *and* $\nu = -\frac{3}{2} + \mu$, $\mu > 0$, *when putting* $r = (p+1)\mu$, *we have*

$$T_1 = \begin{cases} O(h^r), & \text{if } r < 2, \\ O\left(h^2\sqrt{\ln\frac{1}{h}}\right) = O(h^{2-\delta}), \quad 0 < \delta \ll 1, & \text{if } r = 2, \\ O(h^2), & \text{if } r > 2. \end{cases}$$

Next, we estimate bounds on T_2 in (3.2). Consider the quadrature rule on $\triangle_{ij}$,

$$\widehat{\iint_{\triangle_{ij}}} f = \frac{|\triangle_{ij}|}{4}(2f_1 + f_2 + f_3), \tag{4.1}$$

where 1 denotes the right-angled vertex and 2 and 3 the other two vertices of $\triangle_{ij}$. We have following lemma, whose proof is given in Li et al. [7].

Lemma 4.4. *Let* $\triangle_{ij}$ *be a right-angled triangle with the boundary lengths* h_{ij} *and* k_{ij} *forming the right angle satisfying* $k_{ij} \leqslant Ch_{ij}$ *for a positive constant* C, *independent of* h_{ij} *and* k_{ij}. *Then, the quadrature rule in* (4.1) *satisfies*

$$\left|\iint_{\triangle_{ij}} f - \widehat{\iint_{\triangle_{ij}}} f\right| \leqslant Ch_{ij}\sqrt{h_{ij}k_{ij}}|f|_{1,\triangle_{ij}}, \tag{4.2}$$

where $|f|_{k,\triangle_{ij}}$ *denotes the Sobolev norm of* f $\triangle_{ij}$.

From Lemma 4.4 and the Cauchy–Schwarz inequality we have the following lemma (see [7]).

Lemma 4.5. *Let $S_\triangle = \bigcup_{ij} \triangle_{ij}$, and let h_{ij} and k_{ij} be the lengths of for the two sides of $\triangle_{ij}$ forming the right angle. If $h_{ij} \leqslant C k_{ij}$ for a constant $C > 0$, independent of the mesh sizes, then there exists the bound,*

$$\frac{\left|(\iint_{S_\triangle} - \widehat{\iint_{S_\triangle}}) f w\right|}{\overline{\|w\|}_{1,S_\triangle}} \leqslant C \overline{|||\delta|||}_\triangle, \quad w \in V_h^0, \tag{4.3}$$

where $w \in V_h^0$ and

$$\overline{|||\delta|||}_\triangle = \sqrt{\sum_{ij} \left\{ \iint_{\triangle_{ij}} h_{ij}^2 (\tilde{f}_{ij})^2 + \|h_{ij}(D\tilde{f}_{ij})\|_{-1,\triangle_{ij}}^2 \right\}}.$$

Here $\tilde{f}_{ij} = f(\xi_{ij})$ with ξ_{ij} a fixed (but unknown) point in $\triangle_{ij}$, and $Df = \sqrt{f_x^2 + f_y^2}$.

Theorem 4.1. *Let* **A1**–**A4** *be fulfilled. Then, there exists a positive constant C, independent of mesh sizes, such that*

$$T_2 \leqslant C \sum_{\ell=1}^{n} \left\{ \ell^{2\mu(p+1)-3} \times h^{2\mu(p+1)} t_\ell \right\}^{\frac{1}{2}}, \tag{4.4}$$

where $h = \frac{1}{n}$ as given in **A3**. *Moreover, for $\sigma = \frac{1}{2} + \mu$ and $\nu = -\frac{3}{2} + \mu$, $\mu > 0$, we have*

$$T_2 = \begin{cases} O(h^r), & \text{if } r < 1.5, \\ O\left(h^{1.5}\sqrt{\ln \frac{1}{h}}\right) = O(h^{1.5-\delta}), \quad 0 < \delta \ll 1, & \text{if } r = 1.5, \\ O(h^{1.5}), & \text{if } r > 1.5, \end{cases} \tag{4.5}$$

where $r = (p+1)\mu$.

Proof. Choose the difference grids (x_i, y_j) with the distance $d_\ell = O((\ell h)^{p+1})$ to Γ, and let $t_\ell = d_{\ell+1} - d_\ell = O(h(\ell h)^p)$ denote the diameters of $\square_{ij}$ and $\triangle_{ij}$. From Assumption **A4**, the elements $\triangle_{ij} \in S_\triangle$ can be re-ordered according to the distance sequence, d_ℓ, to Γ with $\max\{h_{ij}, k_{ij}\} \leqslant C t_\ell$ and then

$$(D^k \tilde{f}_{ij})^2 = O(d_\ell^{2\nu-2k}) = O(d_\ell^{2\sigma-4-2k}).$$

We have

$$\iint_{\triangle_{ij}} h_{ij}^2 \tilde{f}_{ij}^2 \leqslant C t_\ell^4 d_\ell^{2\nu} = C t_\ell^3 d_\ell^{2\sigma-4} t_\ell \leqslant C \ell^{(2\sigma-1)(p+1)-3} \times h^{(2\sigma-1)(p+1)} t_\ell,$$

$$\|h_{ij}(D\tilde{f}_{ij})\|_{-1,\triangle_{ij}}^2 \leqslant C \ell^{(2\sigma-1)(p+1)-3} \times h^{(2\sigma-1)(p+1)} t_\ell.$$

Hence, we have from Lemma 4.5

$$T_2 \leqslant C \overline{|||\delta|||}_\triangle \leqslant \left\{ \sum_{ij \in I_S} \left(\iint_{\triangle_{ij}} h_{ij}^2 \tilde{f}_{ij}^2 + \|h_{ij} D\tilde{f}_{ij}\|_{-1,\triangle_{ij}}^2 \right) \right\}^{\frac{1}{2}}$$

$$\leqslant C \left\{ \sum_{\ell=1}^{n} \ell^{(2\sigma-1)(p+1)-3} \times h^{(2\sigma-1)(p+1)} t_\ell \right\}^{\frac{1}{2}} \leqslant C \left\{ \sum_{\ell=1}^{n} \ell^{2\mu(p+1)-3} \times h^{2\mu(p+1)} t_\ell \right\}^{\frac{1}{2}}.$$

This is (4.4), and Eq. (4.5) follows from $r = (p+1)\mu$ and $\sum_{\ell=1}^{n} t_\ell \leqslant C$. It completes the proof of Theorem 4.1. □

5. Bounds on T

In this section, we derive some bounds on T in (3.2). To achieve this, we first note

$$\left(\widehat{\iint_{S_\Box}} - \iint_{S_\Box}\right)\nabla u \nabla v\, ds = \left(\widehat{\iint_{S_\Box}} - \iint_{S_\Box}\right)\left\{(u_I)_x v_x + (u_I)_y v_y\right\}$$
$$= \sum_{ij\in I_S}\left(\widehat{\iint_{\Box_{ij}}} - \iint_{\Box_{ij}}\right)\left\{(u_I)_x v_x + (u_I)_y v_y\right\}. \tag{5.1}$$

The present analysis has two differences from that in [6]: (1) The Dirichlet boundary condition is not imposed on $\partial S_\Box \subset S$, where $S_\Box = \bigcup_{ij}\Box_{ij}$. (2) Both grids x_i and y_j along x axis and y axis are chosen to be local refinements. Let us estimate the bounds of the first term on the right hand side of (5.1). After some manipulations, we obtain

$$\iint_{\Box_{ij}} (u_I)_x v_x = \frac{k_j}{3h_i}\left\{(u_4-u_3)(v_4-v_3) + (u_2-u_1)(v_2-v_1)\right.$$
$$\left. + \frac{1}{2}(u_4-u_3)(v_2-v_1) + \frac{1}{2}(u_2-u_1)(v_4-v_3)\right\},$$

and

$$\widehat{\iint_{\Box_{ij}}} (u_I)_x v_x = \frac{k_j}{2h_i}\left\{(u_4-u_3)(v_4-v_3) + (u_2-u_1)(v_2-v_1)\right\},$$

where subscripts 1, 2, 3 and 4 represent grids (i,j), $(i+1,j)$, $(i,j+1)$ and $(i+1,j+1)$ respectively. Hence we have

$$\left(\widehat{\iint_{\Box_{ij}}} - \iint_{\Box_{ij}}\right)(u_I)_x v_x = \frac{k_j}{6h_i}(u_1+u_4-u_2-u_3)(v_1+v_4-v_2-v_3). \tag{5.2}$$

Using the Taylor expansion we see that u_4 can be expanded at the center O at $(i+\frac{1}{2}, j+\frac{1}{2})$ to yield

$$u_4 := u(i+1,j+1) = u_o + \left(\frac{h_i}{2}\frac{\partial}{\partial x} + \frac{k_j}{2}\frac{\partial}{\partial y}\right)u_o + \frac{1}{2}\left(\frac{h_i}{2}\frac{\partial}{\partial x} + \frac{k_j}{2}\frac{\partial}{\partial y}\right)^2 u_o$$
$$+ \frac{1}{3!}\left(\frac{h_i}{2}\frac{\partial}{\partial x} + \frac{k_j}{2}\frac{\partial}{\partial y}\right)^3 u_o + \frac{1}{4!}\left(\frac{h_i}{2}\frac{\partial}{\partial x} + \frac{k_j}{2}\frac{\partial}{\partial y}\right)^4 \tilde{u},$$

where $\tilde{u} = u(\xi,\eta)$ for some $(\xi,\eta)\in\Box_{ij}$. The other terms, u_k, $k=1,2,3$, can be derived similarly. Using these expansions we have

$$u_1+u_4-u_2-u_3 = h_i k_j (u_{xy})_o + \frac{k_j^4}{4\cdot 4!}\tilde{u}_{yyyy} + b_1 h_i k_j^3 \tilde{u}_{xyyy}$$
$$+ b_2 h_i^2 k_j^2 \tilde{u}_{xxyy} + b_3 h_i^3 k_j \tilde{u}_{xxxy} + b_\ell h_i^4 \tilde{u}_{xxxx},$$

where b_ℓ's are constants independent of h_i and k_j. Substituting the above equation into (5.2) we have

$$\left(\widehat{\iint_{S_\Box}} - \iint_{S_\Box}\right)(u_I)_x v_x = \sum_{ij\in I_S}\left(\widehat{\iint_{\Box_{ij}}} - \iint_{\Box_{ij}}\right)(u_I)_x v_x$$
$$= \sum_{ij\in I_S}\frac{k_j}{6h_i}\left\{h_i k_j (u_{xy})_o + \frac{k_j^4}{4\cdot 4!}\tilde{u}_{yyyy} + \sum_{\ell=1}^{4} b_\ell h_i^\ell k_j^{4-\ell}\tilde{u}_{x^\ell y^{4-\ell}}\right\}$$
$$\times (v_1+v_4-v_2-v_3). \tag{5.3}$$

We will first estimate the bound on the first term on the right-hand side of (5.3). This is given in the following lemma.

Lemma 5.1. *Let* **A4–A6** *hold. The there exists the error bound,*

$$\left|\sum_{ij\in I_S} k_j^2 (u_{xy})_o (v_1+v_4-v_2-v_3)\right| \leqslant Ch^{\frac{3}{2}}\,\overline{\|v\|}_1, \quad \forall v\in V_h^0. \tag{5.4}$$

Lemma 5.2. *Let* **A4–A6** *hold. Then we have*

$$\left|\left(\widehat{\iint_{S_\square}} - \iint_{S_\square}\right)(u_I)_x v_x\right| \leqslant C\left\{h^{1.5} + h^3\|\tilde u_{xyyy}\|_{0,S} + T_0\right\} \times \overline{\|v\|}_1, \quad v\in V_h^0,$$

where

$$T_0 = \sqrt{\sum_{ij\in I_S}\left\|h_i^3\tilde u_{xxxx}\right\|^2_{0,\square_{ij}}} + h\sqrt{\sum_{ij\in I_S}\left\|h_i^2\tilde u_{xxxy}\right\|^2_{0,\square_{ij}}} + h^2\sqrt{\sum_{ij\in I_S}\left\|h_i\tilde u_{xxyy}\right\|^2_{0,\square_{ij}}} + h^{1.5}\sqrt{\sum_{ij\in I_S}\left\|\frac{k_j^{2.5}}{h_i}\tilde u_{yyyy}\right\|^2_{0,\square_{ij}}} \tag{5.5}$$

and $\tilde u = u(\xi,\eta)$ *form some* $(\xi,\eta)\in\square_{ij}$.

The proof Lemmas 5.1 and 5.2 is provided in [7]. Similarly, we can prove the following lemma (see [7]).

Lemma 5.3. *Let* **A4–A6** *hold. For* $\sigma = \frac{1}{2}+\mu$ *and* $\nu = -\frac{3}{2}+\mu$, $\mu>0$, *putting* $r=(p+1)\mu\leqslant 1.5$, *we have*

$$\left|\left(\widehat{\iint_{S_\square}} - \iint_{S_\square}\right)(u_I)_y v_y\right| \leqslant C\left\{h^r + h^3\|\tilde u_{xyyy}\|_{0,S} + T_0^*\right\} \times \overline{\|v\|}_1, \quad v\in V_h^0, \tag{5.6}$$

where

$$T_0^* = \sqrt{\sum_{ij\in I_S}\left\|h_i^3\tilde u_{xxxx}\right\|^2_{0,\square_{ij}}} + h\sqrt{\sum_{ij\in I_S}\left\|h_i^2\tilde u_{xxxy}\right\|^2_{0,\square_{ij}}} + h^2\sqrt{\sum_{ij\in I_S}\left\|h_i\tilde u_{xxyy}\right\|^2_{0,\square_{ij}}} + \sqrt{\sum_{ij\in I_S}\left\|\frac{h_i^4}{k_j}\tilde u_{yyyy}\right\|^2_{0,\square_{ij}}} \tag{5.7}$$

and $\tilde u = u(\xi,\eta)$ *for some* $(\xi,\eta)\in\square_{ij}$.

Theorem 5.1. *Let* **A4–A6** *hold, and assume that the exact solution, u, is four-times continuously differentiable on S, except the x-derivatives of u at* $x=0$. *Let* $\sigma=\frac{1}{2}+\mu$, $\mu>0$ *and* $r=(p+1)\mu\leqslant 1.5$. *Then, we have*

$$T = \sup_{v\in V_h^0}\frac{1}{\overline{\|v\|}_1}\left|\left(\widehat{\iint_{S_\square}} - \iint_{S_\square}\right)\nabla u_I\nabla v\right| = \mathrm{O}\left(h^r\right). \tag{5.8}$$

Proof. We have from Lemmas 5.2 and 5.3,

$$\left|\left(\widehat{\iint_{S_\square}} - \iint_{S_\square}\right)\left\{(u_I)_x v_x + (u_I)_y v_y\right\}\right| \leqslant C\left\{h^r + h^{1.5} + h^3\|\tilde u_{xyyy}\|_{0,S} + \bar T_0^*\right\}\overline{\|v\|}_1 \tag{5.9}$$

for $v\in V_h^0$, where

$$\bar{T}_0^* = \sqrt{\sum_{ij\in I_S} \left\| h_i^3 \tilde{u}_{xxxx} \right\|_{0,\square_{ij}}^2} + h\sqrt{\sum_{ij\in I_S} \left\| h_i^2 \tilde{u}_{xxxy} \right\|_{0,\square_{ij}}^2} + h^2\sqrt{\sum_{ij\in I_S} \left\| h_i \tilde{u}_{xxyy} \right\|_{0,\square_{ij}}^2}$$
$$+ h^{1.5}\sqrt{\sum_{ij\in I_S} \left\| \frac{k_j^{2.5}}{h_i} \tilde{u}_{yyyy} \right\|_{0,\square_{ij}}^2} + \sqrt{\sum_{ij\in I_S} \left\| \frac{h_i^4}{k_j} \tilde{u}_{yyyy} \right\|_{0,\square_{ij}}^2}. \tag{5.10}$$

Obviously, we have from $\sigma > \frac{1}{2}$,

$$h^r + h^{1.5} + h^3 \|\tilde{u}_{xyyy}\|_{0,S} \leqslant Ch^r. \tag{5.11}$$

It was proved in [6] that

$$\sqrt{\sum_{ij\in I_S} \left\| h_i^3 \tilde{u}_{xxxx} \right\|_{0,\square_{ij}}^2} + \sqrt{\sum_{ij\in I_S} \left\| h_i^2 \tilde{u}_{xxxy} \right\|_{0,\square_{ij}}^2} + h^2\sqrt{\sum_{ij\in I_S} \left\| h_i \tilde{u}_{xxyy} \right\|_{0,\square_{ij}}^2} \leqslant Ch^{1.5}. \tag{5.12}$$

Let us now consider the bounds for the last two terms, denoted by $I^{1/2}$ and $II^{1/2}$ respectively, on the right hand side of (5.10). Since $u_{yyyyy} \in C(S)$, we obtain

$$I := h^3 \sum_{ij\in I_S} \left\| \frac{k_j^{2.5}}{h_i} \tilde{u}_{yyyy} \right\|_{0,\square_{ij}}^2 = h^3 \sum_{ij\in I_S} \iint_{\square_{ij}} \frac{k_j^5}{h_i^2} \tilde{u}_{yyyyy}^2 \leqslant Ch^3 \sum_{ij\in I_S} \frac{k_j^6}{h_i}. \tag{5.13}$$

Since

$$\sum_{i=1}^N \frac{1}{h_i} \leqslant C \frac{1}{i^p h^{p+1}},$$

we have

$$\sum_{i=1}^N \frac{1}{h_i} \leqslant C \frac{1}{h^2} \ln \frac{1}{h} \quad \text{for } p = 1$$

and

$$\sum_{i=1}^N \frac{1}{h_i} \leqslant C \frac{1}{h^2} \quad \text{for } p \neq 1.$$

Then, we have

$$I \leqslant Ch^3 \left(\sum_{j=1}^N h_j^6 \right) \left(\sum_{i=1}^N \frac{1}{h_i} \right) \leqslant Ch^8 \frac{1}{h^2} \ln \frac{1}{h} \leqslant Ch^6 \ln \frac{1}{h}. \tag{5.14}$$

Moreover, we have

$$II := \sum_{ij\in I_S} \left\| \frac{h_i^4}{k_j} \tilde{u}_{xxxx} \right\|_{0,\square_{ij}}^2 \leqslant \sum_{ij\in I_S} \iint_{\square_{ij}} \frac{h_i^8}{k_j^2} \tilde{u}_{xxxx}^2$$
$$\leqslant \sum_{ij\in I_S} \frac{h_i^9}{k_j} \tilde{u}_{xxxx}^2 \leqslant C \sum_{ij\in I_S} \frac{h_i^2}{k_j} h_i^7 x_i^{2\sigma-8} \leqslant Ch^2 \left(\sum_{i=1}^N h_i^7 x_i^{2\sigma-8} \right) \left(\sum_{j=1}^N \frac{1}{k_j} \right). \tag{5.15}$$

Next, for $r \leqslant 1.5$ we obtain

$$\sum_{i=1}^N h_i^7 x_i^{2\sigma-8} \leqslant C \sum_{i=1}^N i^{2\sigma-1)(p+1)-7} h^{(2\sigma-1)(p+1)} \leqslant C \sum_{i=1}^N i^{2r-7} h^{2r} \leqslant C \frac{\ln N}{N^3} h^{2r}. \tag{5.16}$$

Since $\sum_{j=1}^N \frac{1}{k_j} \leqslant C \frac{1}{h^2} (\ln \frac{1}{h})$, we obtain from (5.15) and (5.16),

$$II \leqslant Ch^{3+2r}\left(\ln\frac{1}{h}\right)^2. \tag{5.17}$$

Combining (5.9)–(5.12), (5.14) and (5.17) gives the desired result (5.8). This completes the proof of Theorem 5.1. □

Remark 5.1. In [7], the FDM is extended to other elliptic equations with Neumann and Robin boundary conditions. Numerical experiments, displaying a computed convergence rate of order $O(h^{1.85})$, are also presented in [7] to support the theoretical $O(h^{1.5})$-order superconvergence rate obtained in this paper.

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References

[1] J.H. Bramble, B.E. Hubbard, On the formulation of finite difference analogies of the Dirichlet problem for Poisson's equation, Numer. Math. 4 (1962) 313–327.

[2] C.G. Ciarlet, Basic error estimates for elliptic problems, in: P.G. Ciarlet, J.L. Lions (Eds.), Finite Element Methods (Part I), North-Holland, Amsterdam, 1991.

[3] Q. Fang, Convergence of finite difference methods for convection–diffusion problems with singular solutions, J. Comput. Appl. Math. 152 (2003) 119–131.

[4] M. Krizek, P. Neittaanmaki, On a global-superconvergence of the gradient of linear triangular elements, J. Comput. Appl. Math. 18 (1987) 221–233.

[5] Z.C. Li, Combined Methods for Elliptic Equations with Singularities, Interfaces and Infinities, Kluwer Academic Publishers, Dordrecht, Boston, London, 1998.

[6] Z.C. Li, H.Y. Hu, Q. Fang, T. Yamamoto, Superconvergence of solution derivatives for the Shortley–Weller difference approximation of Poisson's equation, Part II. Singularity problems, Numer. Funct. Anal. Optimiz. 24 (3–4) (2003) 195–221.

[7] Z.C. Li, H.Y. Hu, S. Wang, Q. Fang, Superconvergence of solution derivatives for the Shortley–Weller difference approximation of Poisson's equation, Part III. Singularity problems on polygons, Technical report, Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, Taiwan, 2005.

[8] Z.C. Li, S. Wang, The finite volume method and application in combinations, J. Comp. Appl. Math. 106 (1999) 21–53.

[9] Z.C. Li, T. Yamamoto, Q. Fang, Superconvergence of solution derivatives for the Shortley–Weller difference approximation of Poisson's equation, Part I. Smoothness problems, J. Comp. Appl. Math. 151 (2) (2003) 307–333.

[10] G. Strang, G.J. Fix, An Analysis of the Finite Element Method, Prentice-Hall, Inc., 1973.

[11] T. Yamamoto, Inversion formulas for tridiagonal matrices with applications to boundary value problems, Numer. Funct. Anal. Optimiz. 22 (2001) 357–385.

[12] T. Yamamoto, Convergence of consistent and inconsistent finite difference schemes and an acceleration technique, J. Comp. Appl. Math. 140 (2002) 849–866.

[13] T. Yamamoto, Q. Fang, X. Chen, Superconvergence and nonsuperconvergence of the Shortley–Weller approximations for Dirichlet problems, Numer. Funct. Anal. Optimiz. 22 (2001) 455–470.

[14] K. Yoshida, T. Tsuchiya, Recovered derivatives for the Shortley–Weller finite difference approximation, Information 4 (3) (2001) 267–277.