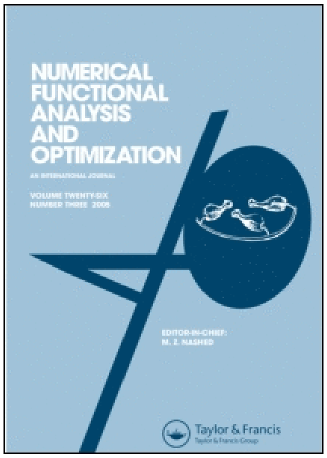

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SUPERCONVERGENCE OF SOLUTION DERIVATIVES OF THE SHORTLEY–WELLER DIFFERENCE APPROXIMATION TO ELLIPTIC EQUATIONS WITH SINGULARITIES INVOLVING THE MIXED TYPE OF BOUNDARY CONDITIONS

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□ *This paper presents a superconvergence analysis for the Shortley–Weller finite difference approximation of second-order self-adjoint elliptic equations with unbounded derivatives on a polygonal domain with the mixed type of boundary conditions. In this analysis, we first formulate the method as a special finite element/volume method. We then analyze the convergence of the method in a finite element framework. An $O(h^{1.5})$-order superconvergence of the solution derivatives in a discrete H^1 norm is obtained. Finally, numerical experiments are provided to support the theoretical convergence rate obtained.*

Keywords Boundary singularity; Finite difference method; Mixed type of boundary conditions; Poisson's equation; Polygonal domains; Shortley–Weller approximation; Solution derivatives; Stretching function; Superconvergence.

AMS Subject Classification 65N10; 65N30.

1. INTRODUCTION

In [7, 8], superconvergence of solution derivatives was studied for the Shortley–Weller finite approximations (simply called FDM) to the Poisson equation with the Dirichlet boundary condition. In this paper, we explore

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the following problem of elliptic equations with the mixed type of the Dirichlet and the Robin boundary conditions,

$$-\Delta u + cu = f(x, y), \quad (x, y) \in S, \tag{1.1}$$

$$u = g(x, y), \quad (x, y) \in \Gamma_D, \tag{1.2}$$

$$u_n + \alpha u = g_R(x, y), \quad (x, y) \in \Gamma_R, \tag{1.3}$$

where S is a polygon, $\Gamma = \Gamma_D \cup \Gamma_R$, and Γ_D and Γ_R are portions of the boundary Γ of S on which the Dirichlet and the Robin conditions are imposed, respectively, and c and α are non-negative smooth functions. In (1.1)–(1.3), the notations are $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ and $u_n = \frac{\partial u}{\partial n}$, and f, g, and g_R are given functions. Note that the Neumann condition is a special case of the Robin condition in (1.3) with $\alpha = 0$. To guarantee that the problem is uniquely solvable, we exclude the case of $c = 0$ and the pure Neumann boundary condition (i.e., $\Gamma_R = \Gamma$ and $\alpha = 0$). Suppose that there exist unbounded derivatives only near boundary Γ_D. The superconvergence rates of orders $O(h^2)$ and $O(h^{1.5})$ of the solution derivatives in a discrete H^1 norm have been established for smooth problems on rectangular and polygonal domains (cf. [7]), respectively. For problems with unbounded derivatives near Γ_D, the local refinements of difference grids nearby should be adapted, the same order superconvergence of solution derivatives can be achieved for rectangular and polygonal domains (see [8, 9]). The above study was made only for Poisson's equation with the Dirichlet boundary condition; this paper pursues the superconvergence for (1.1)–(1.3) by the FDM. Note that in [9], only a brief outline of proof was provided, and no numerical experiments were reported. In this paper, we will provide the detailed proof and the numerical examples on nonrectangular domain with the mixed type of the Dirichlet and the Neumann boundary conditions.

First we formulate (1.1)–(1.3) as a variational problem. Let $L^2(S)$ be the space of square integrable functions on S, and let $H^1(S)$ be the usual Sobolev space. We put $H_0^1 := \{v : v \in H^1(S), v|_{\Gamma_D} = 0\}$. The variational problem corresponding with (1.1)–(1.3) can be expressed by: Find $u \in H^1(S)$ such that

$$a_h(u, v) = f_h(v), \quad \forall v \in H_0^1(S),$$

where the bilinear and linear forms are defined respectively by

$$\begin{aligned} a_h(u, v) &= \iint_S \nabla u \nabla v \, ds + \iint_S cuv \, ds + \int_{\Gamma_R} \alpha uv \, dl, \\ f_h(v) &= \iint_S fv \, ds + \int_{\Gamma_R} g_R v \, dl. \end{aligned} \tag{1.4}$$

This paper is organized as follows. In the next section, we will first present the Shortley–Weller difference scheme as a special finite element method (FEM). In Section 3, an error analysis for the numerical method is presented and the superconvergence of order $O(h^{1.5})$ in a discrete norm is obtained. In Sections 4–5, the proof for lemmas is given. Finally, numerical experiments are presented in the last section to support the theoretical superconvergence results obtained.

2. THE FINITE DIFFERENCE METHODS

A polygonal S may be divided into a mesh containing rectangles and triangles with the mesh lines either parallel to one of the axes or on Γ. In this mesh, all the triangles have at least one side on Γ. Let I be the (double) index set of this mesh and $X = (x_i, y_j)$, $\forall(i,j) \in I$ be the set of mesh nodes on $\overline{S}$. We now split the nodal set X into two disjoint subsets: the set containing nodes in S, denoted by S_h, and that on Γ, denoted by Γ_h. The index subsets of I corresponding with S_h and Γ_h are denoted by I_S and I_Γ, respectively. For any feasible indices i and j, let $h_i = x_{i+1} - x_i$ and $k_j = y_{j+1} - y_j$ be the step sizes along the two directions, respectively. We put $h = \max_{i,j}\{h_i, k_j\}$. In what follows, we use $\square_{ij}$ and $\triangle_{ij}$ to denote respectively the rectangular and triangular element associated with (i,j) as shown in Figure 1. For these elements, we have

$$S = S_\square \cup S_\triangle := (\cup_{ij}\square_{ij}) \cup (\cup_{ij}\triangle_{ij}). \tag{2.1}$$

As constructed, all elements in $S_\triangle$ are right-angled triangles and located near the boundary Γ. Therefore, the two nodes other than (x_i, y_j) of $\triangle_{ij} \in S_\triangle$ are in Γ_h. Obviously, the total number of triangles is much less than the number of rectangles in this mesh. As shown in [5], the conventional finite difference method can be formulated as a special finite element method

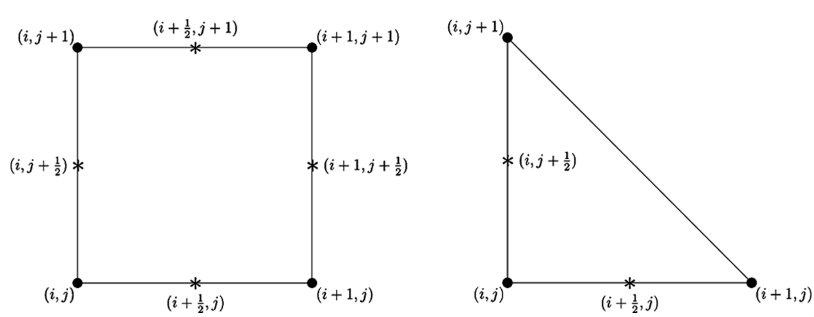

FIGURE 1 A rectangle $\square_{ij}$ and a $\triangle_{ij}$.

using piecewise bilinear and linear interpolating functions, $v(x,y)$, on $\Box_{ij}$ and $\triangle_{ij}$ defined respectively by,

$$v(x,y) = \frac{1}{h_i\,k_j}\{(x_{i+1}-x)(y_{j+1}-y)v_{i,j} + (x-x_i)(y_{j+1}-y)v_{i+1,j} + (x_{i+1}-x)(y-y_j)v_{i,j+1} + (x-x_i)(y-y_j)v_{i+1,j+1}\}, \quad (x,y)\in\Box_{ij}, \tag{2.2}$$

and

$$v(x,y) = v_{i,j} + \frac{(x-x_i)}{h_i}(v_{i+1,j}-v_{i,j}) + \frac{(y-y_j)}{k_j}(v_{i,j+1}-v_{i,j}), \quad (x,y)\in\triangle_{ij}, \tag{2.3}$$

where $v_{k,\ell}$ denotes the nodal value of v at (x_k, y_ℓ). Let $V_h \subseteq H^1(S)$ denote a finite dimensional space of the piecewise bilinear and linear functions v of (2.2) and (2.3) satisfying (1.2), and we denote by V_h^0 the subset of V_h satisfying $v = 0$ on Γ. The FDM with the quadrature approximations to the line and area integrals are defined by: Find $u_h \in V_h$ such that

$$\hat{a}_h(u_h, v) = \hat{f}_h(v), \quad \forall v \in V_h^0, \tag{2.4}$$

where

$$\begin{aligned}\hat{a}_h(u,v) &= \widehat{\iint_S} \nabla u \nabla v\, ds + \widehat{\iint_S} cuv\, ds + \widehat{\int}_{\Gamma_R} \alpha uv\, dl \\ &= \sum_{ij\in I_S}\left[\widehat{\iint}_{\Box_{ij}} \nabla u\nabla v\, ds + \widehat{\iint}_{\triangle_{ij}} \nabla u \nabla v\, ds\right] \\ &\quad + \sum_{ij\in I_S}\left[\widehat{\iint}_{\Box_{ij}} cuv\, ds + \widehat{\iint}_{\triangle_{ij}} cuv\, ds\right] + \sum_i \widehat{\int}_{(\Gamma_R)_i} \alpha uv\, dl, \end{aligned}\tag{2.5}$$

$$\begin{aligned}\hat{f}_h(v) &= \widehat{\iint_S} fv\, ds + \widehat{\int}_{\Gamma_R} g_R v\, dl \\ &= \sum_{ij\in I_S}\left[\widehat{\iint}_{\Box_{ij}} fv\, ds + \widehat{\iint}_{\triangle_{ij}} fv\, ds\right] + \sum_i \widehat{\int}_{(\Gamma_R)_i} g_R v\, dl, \end{aligned}\tag{2.6}$$

where $\Gamma_R = \cup_i(\Gamma_R)_i$, and $(i,j)\in I_S$ means the grids $(x_i, y_j) = (i,j) \in S_h$. The approximate integrals in (2.5) and (2.6) over rectangles $\Box_{ij}$ are evaluated by the following quadrature rules

$$\widehat{\iint_{\square_{ij}}} \nabla u \nabla v \, ds = \widehat{\iint_{\square_{ij}}} u_x v_x \, ds + \widehat{\iint_{\square_{ij}}} u_y v_y \, ds, \tag{2.7}$$

$$\begin{aligned} \widehat{\iint_{\square_{ij}}} u_x v_x \, ds = \frac{h_i \, k_j}{2} \bigg[& u_x\left(i + \frac{1}{2}, j\right) v_x\left(i + \frac{1}{2}, j\right) \\ & + u_x\left(i + \frac{1}{2}, j+1\right) v_x\left(i + \frac{1}{2}, j+1\right)\bigg], \end{aligned} \tag{2.8}$$

$$\begin{aligned} \widehat{\iint_{\square_{ij}}} u_y v_y \, ds = \frac{h_i \, k_j}{2} \bigg[& u_y\left(i, j + \frac{1}{2}\right) v_y\left(i, j + \frac{1}{2}\right) \\ & + u_y\left(i+1, j + \frac{1}{2}\right) v_y\left(i+1, j + \frac{1}{2}\right)\bigg], \end{aligned} \tag{2.9}$$

$$\begin{aligned} \widehat{\iint_{\square_{ij}}} cuv \, ds = \frac{h_i \, k_j}{4} [& (cu)_{ij} v_{ij} + (cu)_{i+1,j} v_{i+1,j} + (cu)_{i,j+1} u_{i,j+1} \\ & + (cu)_{i+1,j+1} \, v_{i+1,j+1}], \end{aligned} \tag{2.10}$$

$$\widehat{\iint_{\square_{ij}}} fv \, ds = \frac{h_i k_j}{4} \left[f_{ij} v_{ij} + f_{i+1,j} v_{i+1,j} + f_{i,j+1} v_{i,j+1} + f_{i+1,j+1} v_{i+1,j+1} \right], \tag{2.11}$$

where $w_{ij} = w(x_i, y_j)$ for any w and ij and

$$u_x\left(i + \frac{1}{2}, j\right) = \frac{u_{i+1,j} - u_{ij}}{h_i}, \quad u_y\left(i, j + \frac{1}{2}\right) = \frac{u_{i,j+1} - u_{ij}}{k_j}$$

with the mesh nodes being defined in Figure 1. Similarly, the integrals over a triangle $\triangle_{ij}$ used in (2.5) and (2.6) are approximated by

$$\begin{aligned} \widehat{\iint_{\triangle_{ij}}} \nabla u \nabla v \, ds = \frac{h_i \, k_j}{2} \bigg[& u_x\left(i + \frac{1}{2}, j\right) v_x\left(i + \frac{1}{2}, j\right) \\ & + u_y\left(i, j + \frac{1}{2}\right) v_y\left(i, j + \frac{1}{2}\right)\bigg], \end{aligned} \tag{2.12}$$

$$\widehat{\iint_{\triangle_{ij}}} cuv \, ds = \frac{h_i k_j}{8} \left[2(cu)_{ij} v_{ij} + (cu)_{i+1,j} v_{i+1,j} + (cu)_{i,j+1} v_{i,j+1} \right], \tag{2.13}$$

$$\widehat{\iint_{\triangle_{ij}}} fv \, ds = \frac{h_i k_j}{8} \left[2 f_{ij} v_{ij} + f_{i+1,j} v_{i+1,j} + f_{i,j+1} v_{i,j+1} \right], \tag{2.14}$$

$$\widehat{\int_{\Gamma_R}} \alpha uv = \sum_i \widehat{\int_{(\Gamma_R)_i}} \alpha uv, \quad \widehat{\int_{\Gamma_R}} g_R v = \sum_i \widehat{\int_{(\Gamma_R)_i}} g_R v. \tag{2.15}$$

From (2.4) and (2.7)–(2.15), the traditional Shortley–Weller difference scheme centered at the grid $(i, j) \in S_h$ is given by

$$
\begin{aligned}
&-\frac{(k_{j-1}+k_j)}{2h_i}(u_{i+1,j}-u_{i,j}) - \frac{(k_{j-1}+k_j)}{2h_{i-1}}(u_{i-1,j}-u_{i,j}) \\
&\quad - \frac{(h_{i-1}+h_i)}{2k_j}(u_{i,j+1}-u_{i,j}) - \frac{(h_{i-1}+h_i)}{2k_{j-1}}(u_{i,j-1}-u_{i,j}) \\
&\quad + \frac{(h_{i-1}+h_i)(k_{j-1}+k_j)}{4} c_{i,j} u_{i,j} \\
&\quad\quad = \frac{(h_{i-1}+h_i)(k_{j-1}+k_j)}{4} f_{i,j}.
\end{aligned}
$$

Dividing both sides of above equation by $\frac{(h_{i-1}+h_i)(k_{j-1}+k_j)}{4}$ gives the Shortley–Weller approximation to the equation. For the integrals on boundary in (2.15), we choose the trapezoidal rule,

$$\widehat{\int}_{\overline{12}} f = \frac{|\overline{12}|}{2}(f_1 + f_2),$$

where 1 and 2 denote the two end points and $|\overline{12}|$ denotes the length of the segment $\overline{12}$.

Based on partition (2.1), the FDM can also be interpreted as the finite volume method (FVM): To seek u_h such that

$$-\widehat{\int}_{\partial S_{ij}\backslash\Gamma} \frac{\partial u_h}{\partial n} + \widehat{\iint}_{S_{ij}} cu_h + \widehat{\int}_{\Gamma_R \cap S_{ij}} \alpha u_h = \widehat{\iint}_{S_{ij}} f + \widehat{\int}_{\Gamma_N \cap S_{ij}} g_N + \widehat{\int}_{\Gamma_R \cap S_{ij}} g_R, \tag{2.16}$$

Eq. (2.16) may greatly simplify the formulation of difference equations of the FDM.

For the discretization of the problem on Γ_R, we have the difference scheme as follows. Taking Figure 2 with the Robin condition as an example, we have

$$-\widehat{\int}_{\overline{AB}\cup\overline{BC}} \frac{\partial u_h}{\partial n} + \widehat{\iint}_{S_{ij}} cu_h + \widehat{\int}_{\overline{OA}\cup\overline{OC}} \alpha u_h = \widehat{\iint}_{S_{ij}} f + \widehat{\int}_{\overline{OA}\cup\overline{OC}} g_R. \tag{2.17}$$

We can obtain the following equation easily,

$$
\begin{aligned}
&-\frac{1}{2}\left\{\frac{u_3-u_0}{k_1}(h_1+h_2) + \frac{u_2-u_0}{h_2}(k_1+k_2)\right\} + (\alpha_0 u_0)|\overline{OA}\cup\overline{OC}| + c_0 u_0 |S_{ij}| \\
&\quad = f_0|S_{ij}| + (g_R)_0|\overline{OA}\cup\overline{OC}|,
\end{aligned}
$$

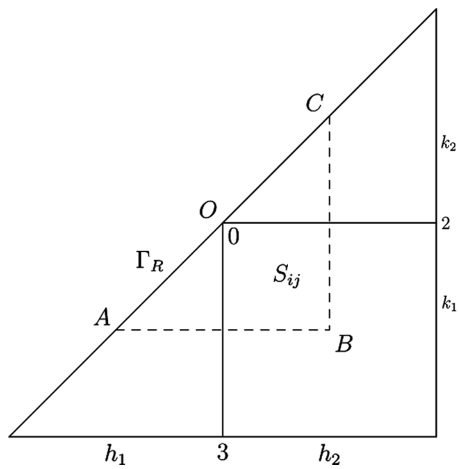

FIGURE 2 Boundary difference equation on Γ_R.

where

$$|S_{ij}| = \frac{1}{8}(h_1 + h_2)(k_1 + k_2), \quad |\overline{OA} \cup \overline{OC}| = \frac{1}{2}\left(\sqrt{h_1^2 + k_1^2} + \sqrt{h_2^2 + k_2^2}\right).$$

The linear algebraic equations from (2.17) are denoted by

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \tag{2.18}$$

where $\boldsymbol{A}$ is symmetric and positive definite, $\boldsymbol{x}$ is the unknown vector consisting of u_{ij}, and $\boldsymbol{b}$ is a known vector. Matrix $\boldsymbol{A}$ is also an M-matrix. Both the maximum principle and the conservative law are retained in the FDM. This is an advantage over the linear FEM. Even for the Dirichlet boundary condition on Γ, the symmetric condition $\frac{\partial u}{\partial n} = 0$ may happen, which is, indeed, the Neumann condition, see the example in the next section.

Note that in FDM, the derivatives u_x and u_y are replaced approximately and straightforwardly by the divided differences. Hence, basic elements in the FDM must be rectangles $\square_{ij}$ and right-angled triangles $\triangle_{ij}$. More general partitions of S into rectangles $\square_{ij}$ and triangles $\triangle_{ij}$ are also possible if we use the ideas of combined methods in [5]: Let S be divided by an interior boundary Γ_0 into several non-overlapped subdomains S_i, $i = 1, 2, \ldots, N$. Each S_i, is further partitioned into rectangles and triangles. We assume that the grid points (i, j) on Γ_0 are common to the meshes on both sides of Γ_0. In this case, our method is conforming because there is no discontinuity across Γ_0 in the piecewise linear and bilinear interpolating functions.

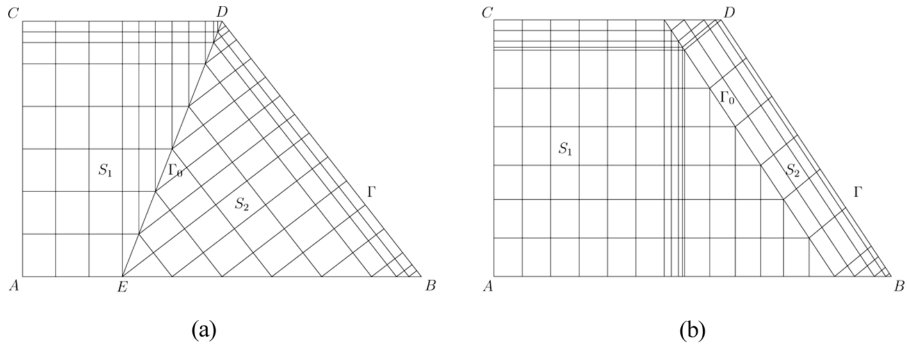

FIGURE 3 Typical local refinements of difference grids. (a) Grids near $\overline{BD} \cup \overline{CD}$; (b) Grids near $\overline{BD}$.

In this section, we will modify the previous numerical method for (1.1) with unbounded derivatives on $\Gamma_U \subseteq \Gamma_D$. This includes a meshing technique and a modification of the difference scheme, as given below.

To mesh S, we follow the following rules:

- We divide S into several subdomains by an interior boundary Γ_0 and mesh each of the subdomain separately.
- For the subdomain containing part of Γ_U, we mesh it using mesh lines parallel or perpendicular to that part of Γ_U. The mesh lines parallel to Γ_U are normally graded according to a rule to be defined.
- Make sure that all the mesh nodes on Γ_0 are common to the meshes on both sides of Γ_0 to avoid the case of nonconformity.

Let us demonstrate this using the subdomain in Figure 3(a), in which the solution derivatives are assumed to be unbounded on two edges, $\overline{CD}$ and $\overline{DB}$. For this case, we split this subdomain into two subsubdomains, named as S_1 and S_2, by $\Gamma_0 = \overline{DE}$, which is chosen such that $\angle CDE \approx \angle EDB$. Then two local Cartesian coordinate systems are used in S_1 and S_2, respectively, and the local nonuniform meshes are used along $\overline{CD}$ and $\overline{BD}$. Figure 3(b) also illustrates a partition used for a problem that has unbounded derivative along the edge $\overline{BD}$. Clearly, a mesh constructed in this way contains mostly rectangular elements. Triangular elements are needed only near Γ and Γ_0.

3. ERROR ANALYSIS OF THE METHOD

In this section, we present the main results of the paper on the superconvergence of the solution of the finite difference method discussed in Section 2. Some detailed proofs of upper bounds for various error terms will be given in Sections 4 and 5 because they are lengthy. This error analysis is carried out using the following discrete H^1 norm:

$$\overline{\|v\|}_1^2 := \overline{\|v\|}_{1,S}^2 = \overline{|v|}_{1,S}^2 + \overline{\|v\|}_{0,S}^2, \quad \forall v \in H^1(S)$$

where $\overline{|v|}_{1,S}$ and $\overline{\|v\|}_{0,S}$ are defined respectively by

$$\overline{|v|}_1^2 = \overline{|v|}_{1,S}^2 = \sum_{ij\in I_S}\left[\widehat{\iint_{\square_{ij}}}(\nabla v)^2 ds + \widehat{\iint_{\triangle_{ij}}}(\nabla v)^2 ds\right],$$

$$\overline{\|v\|}_0^2 = \overline{\|v\|}_{0,S}^2 = \sum_{ij\in I_S}\left[\widehat{\iint_{\square_{ij}}} v^2 ds + \widehat{\iint_{\triangle_{ij}}} v^2 ds\right].$$

Here the quadrature rules, $\widehat{\iint}_{\square_{ij}}(\nabla v)^2\, ds$, $\widehat{\iint}_{\triangle_{ij}}(\nabla v)^2\, ds$, $\widehat{\iint}_{\square_{ij}} v^2\, ds$ and $\widehat{\iint}_{\triangle_{ij}} v^2\, ds$ are given in Section 2.1.

It has been shown in [2], p. 196, that

$$\begin{aligned}\overline{\|u-u_h\|}_1 \le C\Bigg\{&\overline{\|u-u_I\|}_1 + \sup_{w\in V_h^0}\frac{\left|\left(\iint_S - \widehat{\iint}_S\right)\nabla u_I \nabla w\, ds\right|}{\overline{\|w\|}_1}\\ &+ \sup_{w\in V_h^0}\frac{\left|\left(\iint_S - \widehat{\iint}_S\right) cuw\, ds\right|}{\overline{\|w\|}_1} + \sup_{w\in V_h^0} + \frac{\left|\left(\int_{\Gamma_R} - \widehat{\int}_{\Gamma_R}\right)\alpha uw\, dl\right|}{\overline{\|w\|}_1}\\ &+ \sup_{w\in V_h^0}\frac{\left|\left(\iint_S - \widehat{\iint}_S\right) fw\, ds\right|}{\overline{\|w\|}_1} + \sup_{w\in V_h^0}\frac{\left|\left(\int_{\Gamma_R} - \widehat{\int}_{\Gamma_R}\right) g_D w\, dl\right|}{\overline{\|w\|}_1}\Bigg\},\end{aligned} \tag{3.1}$$

where u_I is the V_h-interpolant of the true solution u. Because both u_I and w are linear on $\triangle_{ij}$, it is easy to verify that

$$\iint_{\triangle_{ij}} \nabla u_I \nabla w\, ds - \widehat{\iint_{\triangle_{ij}}} \nabla u_I \nabla w\, ds = 0.$$

Splitting each integral on the right-hand side of (3.1) into one over the union of rectangular elements and another over the union of the triangular elements and using the above equality, we have from (3.1)

$$\begin{aligned}\overline{\|u-u_h\|}_1 \ \le\ C\Bigg\{&\sup_{w\in V_h^0}\frac{\left|\left(\iint_{S_\square} - \widehat{\iint}_{S_\square}\right)\nabla u_I \nabla w\, ds\right|}{\overline{\|w\|}_1}\\ &+ \left(\overline{\|u-u_I\|}_1 + \sup_{w\in V_h^0}\frac{\left|\left(\iint_{S_\square} - \widehat{\iint}_{S_\square}\right) fw\, ds\right|}{\overline{\|w\|}_1}\right)\end{aligned}$$

$$+ \sup_{w \in V_h^0} \frac{\left|\left(\iint_{S_\triangle} - \widehat{\iint}_{S_\triangle}\right) fw\, ds\right|}{\overline{\|w\|}_1}$$

$$+ \left(\sup_{w \in V_h^0} \frac{\left|\left(\iint_{S_\square} - \widehat{\iint}_{S_\square}\right) cuw\, ds\right|}{\overline{\|w\|}_1} + \sup_{w \in V_h^0} \frac{\left|\left(\iint_{S_\triangle} - \widehat{\iint}_{S_\triangle}\right) cuw\, ds\right|}{\overline{\|w\|}_1} \right)$$

$$+ \left(\sup_{w \in V_h^0} \frac{\left|\left(\int_{\Gamma_R} - \widehat{\int}_{\Gamma_R}\right) \alpha uw\, dl\right|}{\overline{\|w\|}_1} + \sup_{w \in V_h^0} \frac{\left|\left(\int_{\Gamma_R} - \widehat{\int}_{\Gamma_R}\right) g_R w\, dl\right|}{\overline{\|w\|}_1} \right) \Bigg\}$$

$$=: T + T_1 + T_2 + T_3 + T_4. \tag{3.2}$$

Naturally, we assume that the unbounded derivatives occur only on the Dirichlet boundary Γ_D. The following assumptions characterize the natures of the singularity in the solution and the right-hand side of the continuous Poisson equation and the finite difference mesh near the boundary $\Gamma_U \subseteq \Gamma_D$.

A1: The derivatives of the solution u close to Γ_U satisfies (cf. [3, 13]):

$$\sup_{0<d(x,y)\ll 1} d^{i-\sigma}(x,y) \left| \frac{\partial^i}{\partial n^i} u(x,y) \right| \le C_1, \quad i = 1, 2, 3, 4,$$

for some constants $\sigma > 0$ and $C_1 > 0$, independent of u, where d denotes the distance between $(x, y) \in S$ and Γ_U, and n denotes the unit vector in the direction from (x, y) to the point on Γ_U closest to (x, y). Moreover, we assume that for $0 < r \ll 1$, u satisfies

$$\sup_{dist(P,Q)\le r} \left| u(P) - u(Q) \right| \le Cr^{\sigma},$$

where P and Q are two arbitrary points in S near Γ_U. In this paper, we assume

$$\sigma = \frac{1}{2} + \mu \tag{3.3}$$

for some $\mu > 0$. This is necessary for the derivative estimates used in the analysis.

A2: The function f satisfies the following properties (cf. [3, 13]):

$$\begin{aligned} &\sup_{0<d\ll 1} d^{i-v}(x,y) \left| \frac{\partial^i}{\partial n^i} f(x,y) \right| \le C_2, \quad i = 0, 1, 2 \\ &\sup_{dist(P,Q)\le r} \| f(P) - f(Q) \| \le Cr^{v} \end{aligned} \tag{3.4}$$

for some constants $v > 0$, $C_2 > 0$, and $0 < r \ll 1$. To make v compatible with (3.3), we assume it satisfies

$$v = -\frac{3}{2} + \mu, \quad \mu > 0.$$

A3: We assume that the difference mesh near Γ_U is constructed locally so that the mesh lines are either parallel or perpendicular to one of the boundary segments in Γ_U. Furthermore, we assume that the mesh lines parallel to one segment of Γ_U are constructed nonuniformly according to the distances from the segment $d_\ell = \psi(\ell h)$, $\ell = 0, 1, \ldots, n$, for a positive integer n, where $h = \frac{1}{n}$ and $\psi(\cdot)$ is the stretching function on $[0, 1]$ defined by

$$\psi(s) = \frac{1}{\int_0^1 z^p(1-z)^p\, dz} \int_0^s z^p(1-z)^p\, dz, \quad s \in [0, 1]$$

with $p > 0$ a (proper) constant (cf. [3, 13]), whose choice will be given in the following theorems.

A triangulation family with the mesh parameter h is said to be regular if for each triangle $\triangle_{ij}$ of the mesh we have if $\frac{\max\{l_{ij}^{(1)}, l_{ij}^{(2)}, l_{ij}^{(3)}\}}{\min\{l_{ij}^{(1)}, l_{ij}^{(2)}, l_{ij}^{(3)}\}} \leq C$ for all $0 < h \leq h_0$, where $l_{ij}^{(k)}$, $k = 1, 2, 3$ denote the lengths of the three sides of $\triangle_{ij}$, h_0 is a positive constant, and C is a positive constant, independent of the mesh sizes. The regularity implies that the interior angles of any $\triangle_{ij}$ are bounded below by a positive constant when $h \to 0^+$. However, for the meshes defined in the previous section, some triangles having at least one side on Γ and Γ_0 may satisfy the above inequality because of the stretching function given in A3. Hence, we make the following assumption.

A4: The family of triangles located on Γ or Γ_0, using the local refinements techniques in A3, is regular.

To make our proof notationally simple, we consider a special case of the above problem that has triangular solution domains of the shape depicted in Figure 4. We simplify Assumptions A1 and A2 as the following:

A5: Suppose that the solution u is sufficiently smooth except[1] for the derivatives with respect to x at $x = 0$, which are unbounded and

[1]This implies that the high-order derivatives with respect to y used are all bounded. However, the derivatives with respect to x are based on (3.5).

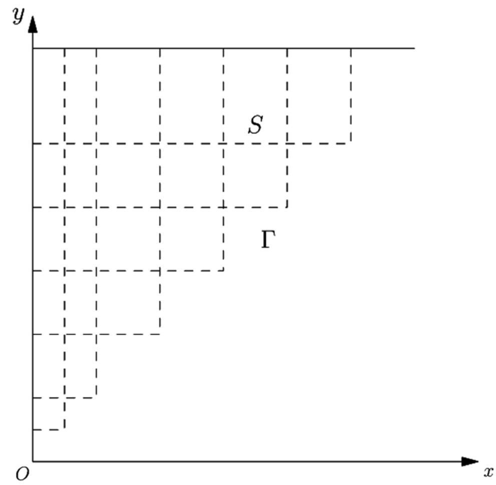

FIGURE 4 Partition of x_i and y_j.

satisfy

$$\begin{aligned} &\sup_{x\in(0,1)} x^{i-\sigma}\left|\frac{\partial^i}{\partial x^i}u(x,y)\right| \le C, \quad i = 1,2,3,4, \\ &\sup_{x\in(0,1)} x^{i-v}\left|\frac{\partial^i}{\partial x^i}f(x,y)\right| \le C, \quad i = 0,1,2, \end{aligned} \tag{3.5}$$

where $\sigma = \frac{1}{2} + \mu$, $v = -\frac{3}{2} + \mu$, and $0 < \mu < \frac{3}{2}$.

Note that Assumptions A1, A2, and A5 are commonly used for problems with unbounded derivatives near the boundary, one popular type of boundary singularities (cf., for example, [3, 4, 8, 12–14]). For this case, the application of the meshing technique in A3 results in a mesh depicted in Figure 4. This is given in the following assumption.

A6: Let both x_i and y_j of partition in S be chosen as in A3; see Figure 4.

Finally, we have the following theorem; the proof of the bounds of T and T_i is deferred to Sections 4 and 5.

Theorem 3.1. *Let $S = S_\square \cup S_\triangle$ and* A1–A6 *be fulfilled. For $\sigma = \frac{1}{2} + \mu$ and $v = -\frac{3}{2} + \mu$, $\mu > 0$, there exists the error bound of the solution u_h from the Shortley–Weller difference approximation*

$$\overline{\|u - u_h\|_1} \le T + T_1 + T_2 + T_3 + T_4 \le Ch^r, \quad r \le 1.5,$$

where $r = (p+1)\mu$, T *and* T_1–T_4 *are defined in* (3.2), *and* C *is a positive constant independent of* h.

4. BOUNDS FOR T_1–T_4

In this section, we will estimate bounds on T_1–T_4 in (3.2). Bounds on T will be established in the next section. The following three lemmas have been established in [7, 8].

Lemma 4.1. *Let* u *be the exact solution to* (1.1)–(1.2), *and let* u_I *be the* V_h-*interpolant of* u. *Then, we have*

$$\overline{\|u-u_I\|_1} = \left\{ \sum_{ij\in I_S} \overline{\|u-u_I\|}^2_{1,\square_{ij}} + \sum_{ij\in I_S} \overline{\|u-u_I\|}^2_{1,\triangle_{ij}} + \sum_i \overline{\|u-u_I\|}^2_{1,(\Gamma_R)_i} \right\}^{\frac{1}{2}},$$

and

$$\begin{aligned}
\overline{|u-u_I|}^2_{1,\square_{ij}} &\le \frac{1}{2}\iint_{\square_{ij}} \{\kappa^2_{1,i}(x)\big(u^2_{xxx}(x,y_j)+u^2_{xxx}(x,y_{j+1})\big) \\
&\qquad + \kappa^2_{2,j}(y)(u^2_{yyy}(x_i,y)+u^2_{yyy}(x_{i+1},y))\},\\
\overline{|u-u_I|}^2_{1,\triangle_{ij}} &\le \iint_{\triangle_{ij}} \{\kappa^2_{1,i}(x)u^2_{xxx}(x,y_j)+\kappa^2_{2,j}(y)u^2_{yyy}(x_i,y)\},\\
\overline{|u-u_I|}^2_{1,(\Gamma_R)_i} &\le \int_{\Gamma_R} \kappa^2_{1,i}(s)u^2_{sss},
\end{aligned}$$

where $u_{x^iy^j} = \frac{\partial^{i+j}u}{\partial x^i\partial y^j}$, *and*

$$\kappa_{1,i}(x) = \begin{cases} \dfrac{1}{2}(x-x_i)^2 & \text{for } x_i \le x \le x_i + \dfrac{h_i}{2},\\[2mm] \dfrac{1}{2}(x_{i+1}-x)^2 & \text{for } x_i + \dfrac{h_i}{2} \le x \le x_{i+1},\end{cases}$$

$$\kappa_{2,j}(y) = \begin{cases} \dfrac{1}{2}(y-y_j)^2 & \text{for } y_j \le y \le y_j + \dfrac{k_j}{2},\\[2mm] \dfrac{1}{2}(y_{j+1}-y)^2 & \text{for } y_j + \dfrac{k_j}{2} \le y \le y_{j+1}.\end{cases}$$

Lemma 4.2. *There exists the bound:*

$$\left|\left(\iint_{S_\square} - \widehat{\iint_{S_\square}}\right) fw\right| \le C\overline{\||f\||_\square} \times \overline{\|w\|_{1,S_\square}}, \quad w \in V_h^0,$$

where

$$\overline{|||f|||}_\square = \sqrt{\sum_{\square_{ij}\in S_\square} \widehat{\iint_{\square_{ij}}} p_{ij}^2 + \|q_{ij}\|^2_{-1,\square_{ij}}},$$

$$p_{ij}^2 = (f_x^2 + f_x^2(x, y_j) + f_x^2(x, y_{j+1}))(x - x_i)^2(x - x_{i+1})^2 + (f_y^2 + f_y^2(x_i, y) + f_y^2(x_{i+1}, y))(y - y_j)^2(y - y_{j+1})^2,$$

$$q_{ij}^2 = (f_{xx}^2 + f_{xx}^2(x, y_j) + f_{xx}^2(x, y_{j+1}))(x - x_i)^2(x - x_{i+1})^2 + (f_{yy}^2 + f_{yy}^2(x_i, y) + f_{yy}^2(x_{i+1}, y))(y - y_j)^2(y - y_{j+1})^2,$$

and

$$\|q_{ij}\|_{-1,\square_{ij}} \leq \sup_{w\in V_h^0} \frac{\iint_{\square_{ij}} q_{ij} w}{\|w\|_{1,\square_{ij}}}.$$

Combining Lemmas 4.1 and 4.2, we have the following lemma.

Lemma 4.3. *Let* A1–A3 *be given. There exists the error bound of* T_1,

$$T_1 \leq C \sum_{\ell=1}^{n} \{\ell^{2\mu(p+1)-5} \times h^{2\mu(p+1)}\}^{\frac{1}{2}},$$

where $h = \frac{1}{N}$, *as given in* A3. *Moreover, for* $\sigma = \frac{1}{2} + \mu$ *and* $v = -\frac{3}{2} + \mu$, $\mu > 0$, *when putting* $r = (p + 1)\mu$, *we have*

$$T_1 = \begin{cases} O(h^r), & \text{if } r < 2, \\ O\left(h^2\sqrt{\ln \dfrac{1}{h}}\right) = O(h^{2-\delta}), \quad 0 < \delta \ll 1, & \text{if } r = 2, \\ O(h^2), & \text{if } r > 2. \end{cases}$$

In the rest of this section, we estimate bounds on T_2 in (3.2). Consider the quadrature rule on $\triangle_{ij}$,

$$\widehat{\iint_{\triangle_{ij}}} f = \frac{|\triangle_{ij}|}{4}(2f_1 + f_2 + f_3), \tag{4.1}$$

where 1 denotes the right-angled vertex and 2 and 3 the other two vertices of $\triangle_{ij}$. We have the following lemma.

Lemma 4.4. *Let $\triangle_{ij}$ be a right-triangled triangle with the boundary lengths h_{ij} and k_{ij} forming the right angle satisfying $k_{ij} \leq Ch_{ij}$ for a positive constant C, independent of h_{ij} and k_{ij}. Then, the quadrature rule in* (4.1) *satisfies*

$$\left| \iint_{\triangle_{ij}} f - \widehat{\iint_{\triangle_{ij}}} f \right| \leq Ch_{ij}\sqrt{h_{ij}k_{ij}}|f|_{1,\triangle_{ij}}, \tag{4.2}$$

where $|f|_{k,\triangle_{ij}}$ denotes the Sobolev norm of $f\triangle_{ij}$.

Proof. Let $\overline{\triangle} = \{(x, y), 0 \leq \bar{x} \leq 1, 0 \leq \bar{x} \leq 1 - \bar{y}\}$ be the reference triangle. For any linear function of the form $\bar{f} = a + b\bar{x} + c\bar{y}$ on $\overline{\triangle}$ with constants a, b, and c, it is easy to verify that

$$\iint_{\overline{\triangle}} \bar{f} = \frac{a}{2} + \frac{b+c}{6} \quad \text{and} \quad \widehat{\iint_{\overline{\triangle}}} \bar{f} = \frac{a}{2} + \frac{b+c}{8}.$$

Hence, the quadrature rule (4.1) is exact only for the constant integrands.

Define an affine transformation $G : (x, y) \to (\bar{x}, \bar{y})$ where $\bar{x} = \frac{x-x_i}{h_{ij}}$ and $\bar{y} = \frac{y-y_j}{k_{ij}}$. Under the transformation G, the triangle $\triangle_{ij}$ is transformed to $\overline{\triangle}$. Let $\bar{f}$ be the image of f under the mapping G. We have

$$\left| \iint_{\triangle_{ij}} f - \widehat{\iint_{\triangle_{ij}}} f \right| = \frac{h_{ij}k_{ij}}{2} \left| \iint_{\overline{\triangle}} \bar{f} - \widehat{\iint_{\overline{\triangle}}} \bar{f} \right| \leq C\,\frac{h_{ij}k_{ij}}{2}|\bar{f}|_{1,\overline{\triangle}}. \tag{4.3}$$

In the last step of (4.3), we used the Bramble–Hilbert lemma in [2], p. 198, because the rule (4.1) is exact for constant. By the inverse transformation G^{-1} of G, we have

$$|\bar{f}|_{1,\overline{\triangle}} \leq C\frac{\max\{h_{ij}, k_{ij}\}}{\sqrt{h_{ij}k_{ij}}}|f|_{1,\triangle_{ij}} \leq C\frac{h_{ij}}{\sqrt{h_{ij}k_{ij}}}|f|_{1,\triangle_{ij}}. \tag{4.4}$$

Combining (4.3) and (4.4) gives the desired result (4.2). This completes the proof of Lemma 4.4. □

Lemma 4.5. *Let $S_\triangle = \cup_{ij}\triangle_{ij}$, and let h_{ij} and k_{ij} be the lengths of for the two sides of $\triangle_{ij}$ forming the right angle. If $h_{ij} \leq Ck_{ij}$ for a constant $C > 0$, independent of the mesh sizes, then there exists the bound,*

$$\frac{|(\iint_{S_\triangle} - \widehat{\iint_{S_\triangle}})fw|}{\overline{\|w\|}_{1,S_\triangle}} \leq C\overline{\||\delta\||}_\triangle, \quad w \in V_h^0, \tag{4.5}$$

where $w \in V_h^0$ *and*

$$\overline{\||\delta\||}_\triangle = \sqrt{\sum_{ij} \left\{ \iint_{\triangle_{ij}} h_{ij}^2 (\tilde{f}_{ij})^2 + \|h_{ij}(D\tilde{f}_{ij})\|^2_{-1,\triangle_{ij}} \right\}}.$$

Here, $\tilde{f}_{ij} = f(\xi_{ij})$ *with* ξ_{ij} *a fixed (but unknown) point in* $\triangle_{ij}$, *and* $Df = \sqrt{f_x^2 + f_y^2}$.

Proof. For any w on, we have $(fw)_x = f_x w + f w_x$ and $(fw)_y = f_y w + f w_y$. From Lemma 4.4, we get

$$\left| \iint_{\triangle_{ij}} fw - \widehat{\iint_{\triangle_{ij}}} fw \right| \le C h_{ij} \sqrt{h_{ij} k_{ij}} \{ |(Df)w|_{0,\triangle_{ij}} + |f(Dw)|_{0,\triangle_{ij}} \},$$

and then from the Cauchy–Schwarz inequality,

$$\begin{aligned} \sum_{ij \in I_S} \left| \iint_{\triangle_{ij}} fw - \widehat{\iint_{\triangle_{ij}}} fw \right| &\le C \sum_{ij \in I_S} \left\{ \left[h_{ij}(D\tilde{f}_{ij}) \sqrt{h_{ij}k_{ij}} \right] \times \left[\tilde{w}_{ij} \sqrt{h_{ij}k_{ij}} \right] \right. \\ &\qquad \left. + \left[(h_{ij}(\tilde{f}_{ij}) \sqrt{h_{ij}k_{ij}} \right] \times \left[(Dw)_{ij} \sqrt{h_{ij}k_{ij}} \right] \right\} \\ &\le C \overline{\||\delta\||}_\triangle \times \overline{\|w\|}_{1,S_\triangle}. \end{aligned}$$

This is (4.5), and we have proved Lemma 4.5. □

From Lemmas 4.4 and 4.5, we have the following theorem:

Theorem 4.6. *Let* A1–A4 *be fulfilled. Then, there exists a positive constant C, independent of mesh sizes, such that*

$$T_2 \le C \sum_{\ell=1}^{n} \{ \ell^{2\mu(p+1)-3} \times h^{2\mu(p+1)} t_\ell \}^{\frac{1}{2}}, \tag{4.6}$$

where $h = \frac{1}{n}$ *as given in* A3. *Moreover, for* $\sigma = \frac{1}{2} + \mu$ *and* $v = -\frac{3}{2} + \mu$, $\mu > 0$, *we have*

$$T_2 = \begin{cases} O(h^r), & \text{if } r < 1.5, \\ O\left(h^{1.5} \sqrt{\ln \frac{1}{h}} \right) = O(h^{1.5-\delta}), \quad 0 < \delta \ll 1, & \text{if } r = 1.5, \\ O(h^{1.5}), & \text{if } r > 1.5, \end{cases} \tag{4.7}$$

where $r = (p+1)\mu$.

Proof. Choose the difference grids (x_i, y_j) with the distance $d_\ell = O((\ell h)^{p+1})$ to Γ, and let $t_\ell = d_{\ell+1} - d_\ell = O(h(\ell h)^p)$ denote the diameters of $\square_{ij}$ and $\triangle_{ij}$. From Assumption A4, the elements $\triangle_{ij} \in S_\triangle$ can be reordered according to the distance sequence, d_ℓ, to Γ with $\max\{h_{ij}, k_{ij}\} \le Ct_\ell$ and then

$$(D^k \tilde{f}_{ij})^2 = O(d_\ell^{2\nu - 2k}) = O(d_\ell^{2\sigma - 4 - 2k}).$$

We have

$$\iint_{\triangle_{ij}} h_{ij}^2 \tilde{f}_{ij}^2 \le Ct_\ell^4 d_\ell^{2\nu} = Ct_\ell^3 d_\ell^{2\sigma-4} t_\ell \le C\ell^{(2\sigma-1)(p+1)-3} \times h^{(2\sigma-1)(p+1)} t_\ell,$$

$$\|h_{ij}(D\tilde{f}_{ij})\|^2_{-1,\triangle_{ij}} \le C\ell^{(2\sigma-1)(p+1)-3} \times h^{(2\sigma-1)(p+1)} t_\ell.$$

Hence, we have

$$\begin{aligned} T_2 &\le C\overline{\||\delta\||}_\triangle \le \left\{ \sum_{ij\in I_S} \left(\iint_{\triangle_{ij}} h_{ij}^2 \tilde{f}_{ij}^2 + \|h_{ij} D\tilde{f}_{ij}\|^2_{-1,\triangle_{ij}} \right) \right\}^{\frac{1}{2}} \\ &\le C \left\{ \sum_{\ell=1}^n \ell^{(2\sigma-1)(p+1)-3} \times h^{(2\sigma-1)(p+1)} t_\ell \right\}^{\frac{1}{2}} \\ &\le C \left\{ \sum_{\ell=1}^n \ell^{2\mu(p+1)-3} \times h^{2\mu(p+1)} t_\ell \right\}^{\frac{1}{2}}. \end{aligned}$$

This is (4.6), and Eq. (4.7) follows from $r = (p+1)\mu$ and $\sum_{\ell=1}^n t_\ell \le C$. This completes the proof of Theorem 4.6. □

Below we provide the bounds of T_3 and T_4. Under A1 and A2, there exist the bounds,

$$|D^k u| \le C|D^k f|, \quad k = 0, 1, \ldots \tag{4.8}$$

For the highly smooth c,[2] we have

$$T_3 = \sup_{w\in V_h^0} \frac{\left|\left(\iint_{S_\square} - \widehat{\iint}_{S_\square}\right) cuw\, ds\right|}{\overline{\|w\|}_1} + \sup_{w\in V_h^0} \frac{\left|\left(\iint_{S_\triangle} - \widehat{\iint}_{S_\triangle}\right) cuw\, ds\right|}{\overline{\|w\|}_1}$$

[2]If the function c is piecewise highly smooth, the same superconvergence can be achieved if its discontinuity boundary is of the difference grids.

$$\leq C\left[\sup_{w\in V_h^0}\frac{\left|\left(\iint_{S_\square}-\widehat{\iint}_{S_\square}\right)fw\,ds\right|}{\overline{\|w\|}_1}+\sup_{w\in V_h^0}\frac{\left|\left(\iint_{S_\triangle}-\widehat{\iint}_{S_\triangle}\right)fw\,ds\right|}{\overline{\|w\|}_1}\right]$$
$$\leq C(T_1+T_2). \tag{4.9}$$

The bound of T_3 is given from Lemma 4.3 and Theorem 4.6. Next, for the bound of T_4, we have the following lemma.

Lemma 4.7. *Suppose $g_R \in H^2(\Gamma_D)$,[3] there exists the bound,*

$$\left|\left(\int_{\Gamma_R}-\widehat{\int}_{\Gamma_R}\right)g_R v\,dl\right| \leq Ch^{1.5}\|g_R\|_{2,\Gamma_R}\overline{\|v\|}_1. \tag{4.10}$$

Proof. Because the central rule is used for $\widehat{\int}_{\Gamma_D}$, we have the bound,

$$\left|\left(\int_{\Gamma_R}-\widehat{\int}_{\Gamma_R}\right)g_R v\,dl\right| \leq C\sum_i h_i^2|g_R v|_{2,(\Gamma_R)_i}. \tag{4.11}$$

By noting the piecewise linear functions u on Γ_R, there exists the bound,

$$\begin{aligned}|g_R v|_{2,(\Gamma_R)_i} &\leq |g_R|_{2,(\Gamma_R)_i}|v|_{0,(\Gamma_R)_i}+2|g_R|_{1,(\Gamma_R)_i}|v|_{1,(\Gamma_R)_i}\\ &\leq |g_R|_{2,(\Gamma_R)_i}|v|_{0,(\Gamma_R)_i}+Ch_i^{-\frac{1}{2}}|g_R v|_{1,(\Gamma_R)_i}\|v\|_{\frac{1}{2},(\Gamma_R)_i}\\ &\leq Ch_i^{-\frac{1}{2}}\|g_R\|_{2,(\Gamma_R)_i}\|v\|_{\frac{1}{2},(\Gamma_R)_i},\end{aligned} \tag{4.12}$$

where we have used the inverse inequality: $|v|_{1,(\Gamma_R)_i} \leq Ch_i^{-\frac{1}{2}}\|v\|_{\frac{1}{2},(\Gamma_R)_i}$. From the Cauchy–Schwarz inequality, we have from (4.11) and (4.12)

$$\begin{aligned}\left|\left(\int_{\Gamma_R}-\widehat{\int}_{\Gamma_R}\right)g_R v\,dl\right| &\leq Ch^{1.5}\|g_R\|_{2,\Gamma_R}\|v\|_{\frac{1}{2},\Gamma_R}\\ &\leq Ch^{1.5}\|g_R\|_{2,\Gamma_R}\|v\|_{1,S} \leq Ch^{1.5}\|g_R\|_{2,\Gamma_R}\overline{\|v\|}_1,\end{aligned} \tag{4.13}$$

where we have used the bounds,

$$\|v\|_{\frac{1}{2},\Gamma_R} \leq C\|v\|_{1,S}, \quad \|v\|_{1,S} \leq C\overline{\|v\|}_1. \tag{4.14}$$

This completes the proof of Lemma 4.7. □

[3]When $g_R = O(u_n)$ near the singular boundary Γ_U, we still have the bound: $|(\int_{\Gamma_R}-\widehat{\int}_{\Gamma_R})g_R v\,dl| \leq Ch^{1.5}\overline{\|v\|}_1$ by following the proof of Lemma 4.8.

Lemma 4.8. *Let* A1, A2, *and* $\alpha \in C^2(\Gamma_R)$ *hold, there exists the bound,*

$$\left|\left(\int_{\Gamma_R} - \widehat{\int}_{\Gamma_R}\right) \alpha uv\, dl\right| \leq Ch^{1.5}\overline{\|v\|}_1. \tag{4.15}$$

Proof. Because $\alpha \in C^2(\Gamma_R)$, we have from the central rule and Lemma 4.7

$$\left|\left(\int_{\Gamma_R} - \widehat{\int}_{\Gamma_R}\right) \alpha uv\, dl\right| \leq C\sum_i h_i^{\frac{3}{2}}|\alpha u|_{2,(\Gamma_R)_i}\|v\|_{\frac{1}{2},(\Gamma_R)_i}$$
$$\leq C\sqrt{\sum_i h_i^3|u|^2_{2,(\Gamma_R)_i}} \times \overline{\|w\|}_1 := C\sqrt{B} \times \overline{\|v\|}_1. \tag{4.16}$$

When the part of Γ_R is far from the singular boundary $\Gamma_U \in \Gamma_D$, $u \in C_2$, and $B = B_{far} = O(h^3)$ to give (4.15) immediately. Below consider the part of Γ_R near Γ_U, and have

$$B = B_{near} = \sum_i h_i^3|u|^2_{2,(\Gamma_R)_i} \leq Ch_i^4 u^2_{ss}((\Gamma_R)_i) \leq Ch_i^4 r_i^{2\sigma-4}, \tag{4.17}$$

where $u_{ss} = \frac{\partial^2 u}{\partial s^2} = O(r_i^{\sigma-2})$. From the local refinement in A3, there exist the bounds,

$$h_i = x_i - x_{i-1} = O(h(ih)^p), \quad r_i = O(x_i) = O((ih)^{p+1}). \tag{4.18}$$

Then we have from (4.17) and (4.18)

$$B_{near} = C\sum_i (h(ih)^p)^4((ih)^{p+1})^{2\sigma-4} \leq Ch^4\sum_i (ih)^{2\sigma(p+1)-4}. \tag{4.19}$$

Because for $\sigma > \frac{1}{2}$ and $p \geq 1$, there exists the bound

$$\sum_i (ih)^{2\sigma(p+1)-4} \leq Ch^{-1}. \tag{4.20}$$

Combining (4.19) and (4.20) gives $B = B_{near} \leq Ch^3$. The desired result (4.15) follows from (4.16), and the proof of Lemma 4.8 is completed. □

From Lemmas 4.7 and 4.8, we have the bound of T_4,

$$T_4 = \left(\sup_{w\in V_h^0} \frac{|(\int_{\Gamma_R} - \widehat{\int}_{\Gamma_R})\alpha uw\, dl|}{\overline{\|w\|}_1} + \sup_{w\in V_h^0} \frac{|(\int_{\Gamma_R} - \widehat{\int}_{\Gamma_R})g_R w\, dl|}{\overline{\|w\|}_1}\right)$$
$$= O(h^{1.5}). \tag{4.21}$$

Remark 4.9. Based on Lemmas 4.7 and 4.8, for the rectangular domain S, only the $O(h^{1.5})$ rate can be obtained, when the Robin condition, or the non-homogeneous Neumann condition $U_n = g_N$, is enforced on Γ_N, where Γ_N is one or two edges of ∂S. However, for the rectangular S, the high superconvergence $O(h^2)$ can also be retained if the homogeneous Neumann condition, $u_n = 0$ on Γ_N, is enforced on Γ_N by noting $T_4 = 0$. Such a conclusion can be confirmed by the following fact. Because $u_n = 0$ implies a symmetry of the elliptic problem, an extended Dirichlet problem can be obtained on a larger rectangle. Hence, the superconvergence $O(h^2)$ can be achieved based on the analysis of [8].

5. BOUNDS ON T

In this section, we derive some bounds on T in (3.2). To achieve this, we first note

$$\left(\widehat{\iint_{S_\square}} - \iint_{S_\square}\right) \nabla u \nabla v \, ds = \left(\widehat{\iint_{S_\square}} - \iint_{S_\square}\right) \{(u_I)_x v_x + (u_I)_y v_y\}$$
$$= \sum_{ij \in I_S} \left(\widehat{\iint_{\square_{ij}}} - \iint_{\square_{ij}}\right) \{(u_I)_x v_x + (u_I)_y v_y\}. \tag{5.1}$$

The current analysis has two differences from that in [8]: (1) The Dirichlet boundary condition is not imposed on $\partial S_\square \subset S$, where $S_\square = \cup_{ij} \square_{ij}$; in this paper the Dirichlet or the Robin condition may be imposed on Γ near $\partial S_\square$. (2) Both grids x_i and y_j along x axis and y axis are chosen to be local refinements. Let us estimate the bounds of the first term on the right-hand side of (5.1). After some manipulations, we obtain

$$\iint_{\square_{ij}} (u_I)_x v_x = \frac{k_j}{3h_i} \bigg\{ (u_4 - u_3)(v_4 - v_3) + (u_2 - u_1)(v_2 - v_1)$$
$$+ \frac{1}{2}(u_4 - u_3)(v_2 - v_1) + \frac{1}{2}(u_2 - u_1)(v_4 - v_3) \bigg\},$$

and

$$\widehat{\iint_{\square_{ij}}} (u_I)_x v_x = \frac{k_j}{2h_i} \{(u_4 - u_3)(v_4 - v_3) + (u_2 - u_1)(v_2 - v_1)\},$$

where subscripts 1, 2, 3, and 4 represent grids (i, j), $(i + 1, j)$, $(i, j + 1)$, and $(i + 1, j + 1)$ respectively. Hence we have

$$\left(\widehat{\iint_{\square_{ij}}} - \iint_{\square_{ij}}\right) (u_I)_x v_x = \frac{k_j}{6h_i} (u_1 + u_4 - u_2 - u_3)(v_1 + v_4 - v_2 - v_3). \tag{5.2}$$

Using the Taylor expansion, we see that u_4 can be expanded at the center O at $\left(i+\frac{1}{2}, j+\frac{1}{2}\right)$ to yield

$$u_4 := u(i+1, j+1) = u_o + \left(\frac{h_i}{2}\frac{\partial}{\partial x} + \frac{k_j}{2}\frac{\partial}{\partial y}\right) u_o + \frac{1}{2}\left(\frac{h_i}{2}\frac{\partial}{\partial x} + \frac{k_j}{2}\frac{\partial}{\partial y}\right)^2 u_o + \frac{1}{3!}\left(\frac{h_i}{2}\frac{\partial}{\partial x} + \frac{k_j}{2}\frac{\partial}{\partial y}\right)^3 u_o + \frac{1}{4!}\left(\frac{h_i}{2}\frac{\partial}{\partial x} + \frac{k_j}{2}\frac{\partial}{\partial y}\right)^4 \tilde{u},$$

where $\tilde{u} = u(\xi, \eta)$ for some $(\xi, \eta) \in \square_{ij}$. The other terms, u_k, $k = 1, 2, 3$, can be derived similarly. Using these expansions, we have

$$u_1 + u_4 - u_2 - u_3 = h_i k_j (u_{xy})_o + \frac{k_j^4}{4 \cdot 4!}\tilde{u}_{yyyy} + b_1 h_i k_j^3 \tilde{u}_{xyyy} + b_2 h_i^2 k_j^2 \tilde{u}_{xxyy} + b_3 h_i^3 k_j \tilde{u}_{xxxy} + b_\ell h_i^4 \tilde{u}_{xxxx},$$

where b_ℓ's are constants independent of h_i and k_j. Substituting the above equation into (5.2) we have

$$\begin{aligned}\left(\widehat{\iint_{S_\square}} - \iint_{S_\square}\right)(u_I)_x v_x &= \sum_{ij \in I_S}\left(\widehat{\iint_{\square_{ij}}} - \iint_{\square_{ij}}\right)(u_I)_x v_x \\ &= \sum_{ij \in I_S}\frac{k_j}{6h_i}\left\{h_i k_j (u_{xy})_o + \frac{k_j^4}{4 \cdot 4!}\tilde{u}_{yyyy} + \sum_{\ell=1}^{4} b_\ell h_i^\ell k_j^{4-\ell}\tilde{u}_{x^\ell y^{4-\ell}}\right\}(v_1 + v_4 - v_2 - v_3). \end{aligned} \quad (5.3)$$

We will first estimate the bound on the first term on the right-hand side of (5.3). This is given in the following lemma.

Lemma 5.1. *Let* A4–A6 *hold. The there exists the error bound,*

$$\left|\sum_{ij \in I_S} k_j^2 (u_{xy})_o (v_1 + v_4 - v_2 - v_3)\right| \leq C h^{\frac{3}{2}} \overline{\|v\|}_1, \quad \forall v \in V_h^0. \quad (5.4)$$

Proof. Because no boundary conditions for V are imposed on $S_\square$, we have (cf. [8])

$$\sum_{ij \in I_S} k_j^2 (u_{xy})_o (v_1 + v_4 - v_2 - v_3)$$

$$
\begin{aligned}
&= \sum_{ij\in I_S} \{k_{j+1}^2 (u_{xy})_{i+\frac{1}{2},j+\frac{1}{2}} - k_j^2 (u_{xy})_{i+\frac{1}{2},j-\frac{1}{2}}\}(v_{i+1,j} - v_{ij}) \\
&\quad + \sum_{(i+1,j)\in\Gamma_1 \wedge (i,j)\in\Gamma_1} k_j^2 (u_{xy})_{i+\frac{1}{2},j+\frac{1}{2}} (v_{i+1,j} - v_{i,j}) \\
&= \sum_{ij\in I_S} k_j^2 \{(u_{xy})_{i+\frac{1}{2},j+\frac{1}{2}} - (u_{xy})_{i+\frac{1}{2},j-\frac{1}{2}}\}(v_{i+1,j} - v_{ij}) \\
&\quad + \sum_{ij\in I_S} \{(k_{j+1}^2 - k_j^2)(u_{xy})_{i+\frac{1}{2},j+\frac{1}{2}}\}(v_{i+1,j} - v_{ij}) \\
&\quad + \sum_{(i+1,j)\in\Gamma_1 \wedge (i,j)\in\Gamma_1} k_j^2 (u_{xy})_{i+\frac{1}{2},j+\frac{1}{2}} (v_{i+1,j} - v_{i,j}) \\
&=: T_A + T_B + T_C,
\end{aligned}
\tag{5.5}
$$

where Γ_1 denotes the set of horizontal segments of $\partial\Box_{ij}$, as shown in Figure 5(a). For the first term on the right-hand side of the above equation, we have

$$
\begin{aligned}
|T_A| &= \left| \sum_{ij\in I_S} k_j^2 \{(u_{xy})_{i+\frac{1}{2},j+\frac{1}{2}} - (u_{xy})_{i+\frac{1}{2},j-\frac{1}{2}}\}(v_{i+1,j} - v_i) \right| \\
&= \left| \sum_{ij\in I_S} k_j \frac{k_{j+1} + k_j}{2} (h_i k_j)(\tilde{u}_{xyy})_{i+\frac{1}{2},j} \frac{v_{i+1,j} - v_{ij}}{h_i} \right| \\
&\le Ch^2 \|\tilde{u}_{xyy}\|_{0,S} \overline{\|v\|}_1,
\end{aligned}
\tag{5.6}
$$

in the last step of the above equation, we have also used the Schwarz inequality.

Next, from $k_j = O(j^p h^{p+1})$ in A3 and A5, we have

$$
|k_{j+1} - k_j| \le Ch^{p+1}((j+1)^p - j^p) \le Cph^{p+1} j^{p-1} \le Cj^{p-1} h^{p+1}, \tag{5.7}
$$

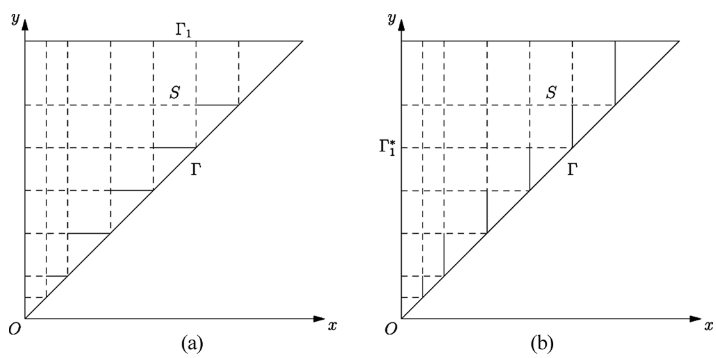

FIGURE 5 The sets (a) Γ_1 and (b) Γ_1^* of $\partial\Box_{ij}$ near Γ highlighted by the solid lines.

and

$$\frac{k_{j+1}+k_j}{k_j} = \frac{(j+1)^p + j^p}{j^p} \le C.$$

Then we obtain

$$\begin{aligned}
|T_B| &= \left| \sum_{ij\in I_S} \{(k_{j+1}^2 - k_j^2)(u_{xy})_{i+\frac{1}{2},j+\frac{1}{2}}\}(v_{i+1,j} - v_{ij}) \right| \\
&= \left| \sum_{ij\in I_S} (k_{j+1} - k_j)\left(\frac{k_{j+1}+k_j}{k_j}\right)(h_i k_j)(u_{xy})_{i+\frac{1}{2},j+\frac{1}{2}} \frac{v_{i+1,j} - v_{ij}}{h_i} \right| \\
&\le CT_D \overline{\|v\|}_1, \qquad (5.8)
\end{aligned}$$

where

$$T_D^2 = \sum_{ij\in I_S} (k_{j+1} - k_j)^2 (h_i k_j)(u_{xy})^2_{i+\frac{1}{2},j+\frac{1}{2}}. \qquad (5.9)$$

From (5.7) we have

$$\begin{aligned}
\sum_{j=1}^{N} (k_{j+1} - k_j)^2 k_j &\le C \sum_{j=1}^{N} j^{2(p-1)} h^{2(p+1)} \times j^p h^{p+1} \\
&= Ch^{3(p+1)} \sum_{j=1}^{N} j^{3p-2} \le CN^{3p-1} h^{3(p+1)} \\
&\le C(Nh)^{3p-1} h^4 \le Ch^4, \qquad (5.10)
\end{aligned}$$

by noting $Nh \le C$.[4] Also from A5 and A6 and $p > 0$

$$\begin{aligned}
\sum_{i=1}^{N} h_i (u_{xy})^2_{i+\frac{1}{2},j+\frac{1}{2}} &\le \sum_{i=1}^{N} h_i x_i^{2(\sigma-1)} \le C \sum_{i=1}^{N} i^p h^{p+1} (ih)^{2(\sigma-1)(p+1)} \\
&= \sum_{i=1}^{N} i^{(2\sigma-1)(p+1)-1} h^{(2\sigma-1)(p+1)} \le CN^{(2\sigma-1)(p+1)} h^{(2\sigma-1)(p+1)} \\
&\le C(Nh)^{(2\sigma-1)(p+1)} \le C. \qquad (5.11)
\end{aligned}$$

Combining (5.8)–(5.11) gives

$$T_D \le Ch^2, \quad |T_B| \le Ch^2 \overline{\|v\|}_1. \qquad (5.12)$$

[4]Strictly speaking, when $3p - 2 = -1$ (i.e., $p = \frac{1}{3}$), the bounds in (5.10) should be modified as $Ch^4 \ln \frac{1}{h}$, and the final bound in (5.4) is also retained.

Moreover,

$$
\begin{aligned}
T_C &= \sum_{(i+1,j)\in\Gamma_1\wedge(i,j)\in\Gamma_1} k_j^2 h_i (u_{xy})_{i+\frac{1}{2},j+\frac{1}{2}} \frac{v_{i+1,j}-v_{i,j}}{h_i} \\
&= \sum_{(i+1,j)\in\Gamma_1\wedge(i,j)\in\Gamma_1} k_j^{\frac{3}{2}} \sqrt{h_i} (u_{xy})_{i+\frac{1}{2},j+\frac{1}{2}} \times \sqrt{h_i k_j} \frac{v_{i+1,j}-v_{i,j}}{h_i} \\
&\le C h^{\frac{3}{2}} \| \tilde{u}_{xy} \|_{0,\Gamma_1} \overline{\|v\|}_1. \qquad (5.13)
\end{aligned}
$$

Combining (5.5), (5.6), (5.12), and (5.13), we finally have

$$
\left| \sum_{ij\in I_S} k_j^2 (u_{xy})_o (v_1+v_4-v_2-v_3) \right| \le C\{ h^{\frac{3}{2}} \| \tilde{u}_{xy} \|_{0,\Gamma_1} + h^2 + h^2 \| \tilde{u}_{xyy} \|_{0,S} \} \overline{\|v\|}_1.
$$

The desired result (5.4) follows from $\|\tilde{u}_{xy}\|_{0,\Gamma_1} \le C$ and $\|\tilde{u}_{xyy}\|_{0,S} \le C$ (see A1). This completes the proof of Lemma 5.1. □

Lemma 5.2. *Let* A4–A6 *hold. Then we have*

$$
\left| \left(\widehat{\iint_{S_\square}} - \iint_{S_\square} \right) (u_I)_x v_x \right| \le C \left\{ h^{1.5} + h^3 \| \tilde{u}_{xyyy} \|_{0,S} + T_0 \right\} \times \overline{\|v\|}_1, \quad v \in V_h^0,
$$

where

$$
\begin{aligned}
T_0 = & \sqrt{\sum_{ij\in I_S} \| h_i^3 \tilde{u}_{xxxx} \|^2_{0,\square_{ij}}} + h \sqrt{\sum_{ij\in I_S} \| h_i^2 \tilde{u}_{xxxy} \|^2_{0,\square_{ij}}} \\
& + h^2 \sqrt{\sum_{ij\in I_S} \| h_i \tilde{u}_{xxyy} \|^2_{0,\square_{ij}}} + h^{1.5} \sqrt{\sum_{ij\in I_S} \left\| \frac{k_j^{2.5}}{h_i} \tilde{u}_{yyyy} \right\|^2_{0,\square_{ij}}} \qquad (5.14)
\end{aligned}
$$

and $\tilde{u} = u(\xi,\eta)$ *form some* $(\xi,\eta) \in \square_{ij}$.

Proof. For the bounds of the terms in (5.3), we have

$$
\begin{aligned}
\left| \sum_{ij\in I_S} \frac{k_j}{h_i} k_j^4 (\tilde{u}_{yyyy})(v_1+v_4-v_2-v_3) \right| &= \left| \sum_{ij\in I_S} (h_i k_j) \frac{k_j^4}{h_i} (\tilde{u}_{yyyy}) \left(\frac{v_4-v_3}{h_i} - \frac{v_2-v_1}{h_i} \right) \right| \\
&\le C h^{\frac{3}{2}} \sqrt{\sum_{ij\in I_S} \left\| \frac{k_j^{2.5}}{h_i} \tilde{u}_{yyyy} \right\|^2_{0,\square_{ij}}} \cdot \overline{\|v\|}_1. \qquad (5.15)
\end{aligned}
$$

Also, we have

$$\left|\sum_{ij\in I_S}\frac{k_j}{h_i}h_i^3k_j(\tilde{u}_{xxxy})(v_1+v_4-v_2-v_3)\right| \le \sum_{ij\in I_S} h_i^3k_j^2|\tilde{u}_{xxxy}|\cdot\left(\left|\frac{v_4-v_3}{h_i}\right|+\left|\frac{v_2-v_1}{h_i}\right|\right)$$
$$\le Ch\sqrt{\sum_{ij\in I_S}\|h_i^2\tilde{u}_{xxxy}\|^2_{0,\square_{ij}}}\cdot\overline{\|v\|}_1. \qquad (5.16)$$

Similarly,

$$\left|\sum_{ij\in I_S}\frac{k_j}{h_i}h_i^4(\tilde{u}_{xxxx})(v_1+v_4-v_2-v_3)\right| \le C\sqrt{\sum_{ij\in I_S}\|h_i^3\tilde{u}_{xxxx}\|^2_{0,\square_{ij}}}\cdot\overline{\|v\|}_1,$$
$$\left|\sum_{ij\in I_S}\frac{k_j}{h_i}h_i^2k_j^2(\tilde{u}_{xxyy})(v_1+v_4-v_2-v_3)\right| \le Ch^2\sqrt{\sum_{ij\in I_S}\|h_i\tilde{u}_{xxyy}\|^2_{0,\square_{ij}}}\cdot\overline{\|v\|}_1,$$
$$\left|\sum_{ij\in I_S}\frac{k_j}{h_i}h_ik_j^3(\tilde{u}_{xyyy})(v_1+v_4-v_2-v_3)\right| \le Ch^3\sqrt{\sum_{ij\in I_S}\|\tilde{u}_{xyyy}\|^2_{0,\square_{ij}}}\cdot\overline{\|v\|}_1$$
$$= Ch^3\|\tilde{u}_{xyyy}\|_{0,S}\cdot\overline{\|v\|}_1. \qquad (5.17)$$

We obtain from Lemma 5.1, (5.3), and (5.15)–(5.17),

$$\left|\left(\widehat{\iint_{S_\square}}-\iint_{S_\square}\right)(u_I)_xv_x\right| \le C\left\{h^{1.5}+h^3\|\tilde{u}_{xyyy}\|_{0,S}+T_0\right\}\times\overline{\|v\|}_1,$$

where T_0 is defined in (5.14). This completes the proof of Lemma 5.2. □

Let us now consider

$$\left(\widehat{\iint_{S_\square}}-\iint_{S_\square}\right)(u_I)_yv_y = \sum_{ij\in I_S}\left(\widehat{\iint_{\square_{ij}}}-\iint_{\square_{ij}}\right)(u_I)_yv_y$$
$$= \sum_{ij\in I_S}\frac{h_i}{6k_j}\left\{h_ik_j(u_{xy})_o+\frac{k_j^4}{4\cdot 4!}\tilde{u}_{yyyy}+\sum_{\ell=1}^{4}b_\ell h_i^\ell k_j^{4-\ell}\tilde{u}_{x^\ell y^{4-\ell}}\right\}$$
$$\times(v_1+v_4-v_2-v_3). \qquad (5.18)$$

We have the following lemma.

Lemma 5.3. *Let* A4–A6 *hold. For* $\sigma=\frac{1}{2}+\mu$ *and* $\nu=-\frac{3}{2}+\mu,\ \mu>0,$ *the following error bound holds:*

$$\left|\sum_{ij\in I_S}h_i^2(u_{xy})_o(v_1+v_4-v_2-v_3)\right| \le Ch^r\overline{\|v\|}_1, \qquad (5.19)$$

where $r=(p+1)\mu\le 1.5$.

Proof. Let the difference grids along x axis are x_i, $i = 0, 1, \dots, N$, and denote $h_i = h_i - h_{i-1}$. We have

$$\begin{aligned}
&\sum_{ij\in I_S} h_i^2 (u_{xy})_o (v_1 + v_4 - v_2 - v_3) \\
&\quad = \sum_{ij\in I_S} \{h_{i+1}^2 (u_{xy})_{i+\frac{1}{2},j+\frac{1}{2}} - h_i^2 (u_{xy})_{i-\frac{1}{2},j+\frac{1}{2}}\}(v_{i,j+1} - v_{ij}) \\
&\qquad + \sum_{(i,j+1)\in\Gamma_1^*\wedge(i,j)\in\Gamma_1^*} h_i^2 (u_{xy})_{i+\frac{1}{2},j+\frac{1}{2}} (v_{i,j+1} - v_{i,j}) \\
&\quad = \sum_{ij\in I_S} h_i^2 \{(u_{xy})_{i+\frac{1}{2},j+\frac{1}{2}} - (u_{xy})_{i-\frac{1}{2},j+\frac{1}{2}}\}(v_{i,j+1} - v_{ij}) \\
&\qquad + \sum_{ij\in I_S} \{(h_{i+1}^2 - h_i^2)(u_{xy})_{i+\frac{1}{2},j+\frac{1}{2}}\}(v_{i,j+1} - v_{ij}) \\
&\qquad + \sum_{(i,j+1)\in\Gamma_1^*\wedge(i,j)\in\Gamma_1^*} h_i^2 (u_{xy})_{i+\frac{1}{2},j+\frac{1}{2}} (v_{i,j+1} - v_{i,j}) \\
&\quad =: T_A^* + T_B^* + T_C^*,
\end{aligned} \tag{5.20}$$

where Γ_1^* denotes the set of vertical segments of $\partial\Box_{ij}$ shown in Figure 5b. For the first term on the right-hand side of the above equation, we have

$$\begin{aligned}
|T_A^*| &= \left| \sum_{ij\in I_S} h_i^2 \{(u_{xy})_{i+\frac{1}{2},j+\frac{1}{2}} - (u_{xy})_{i-\frac{1}{2},j+\frac{1}{2}}\}(v_{i,j+1} - v_i) \right| \\
&= \left| \sum_{ij\in I_S} h_i \frac{h_{i+1} + h_i}{2} (h_i k_j)(\tilde{u}_{xxy})_{i,j+\frac{1}{2}} \frac{v_{i,j+1} - v_{ij}}{k_j} \right| \\
&\le C T_E \overline{\|v\|}_1,
\end{aligned} \tag{5.21}$$

where

$$T_E = \sqrt{\sum_{ij\in I_S} h_i^4 (h_i k_j)(\tilde{u}_{xxy}^2)_{i,j+\frac{1}{2}}}.$$

The bound for T_A^*, which is also different from that in [8], is derived as follows. For the term T_E defined above, we have

$$T_E^2 = \sum_{ij\in I_S} h_i^4 (h_i k_j)(\tilde{u}_{xxy}^2)_{i,j+\frac{1}{2}} = \left\{ \sum_{i=1}^{N} h_i^5 (\tilde{u}_{xxy}^2)_{i,j+\frac{1}{2}} \right\} \left\{ \sum_{j=1}^{N} k_j \right\}.$$

Because $\sum_{j=1}^{N} k_j \leq C,\ u_{xxy} \leq Cx_i^{\sigma-2},\ h_i = \frac{(ih)^{p+1}}{i}$, and $x_i = (ih)^{p+1}$ due to A5 and A6, we obtain

$$T_E^2 \leq C\sum_{i=1}^{N} h_i^5 x_i^{2\sigma-4} = C\sum_{i=1}^{N}(ih)^{2(p+1)} i^{2r-5} \times h^{2r} \leq Ch^{2r},$$

where we have used the facts that $(ih) \leq C$ and $2r = 2(p+1)\mu = 2(2\sigma-3)(p+1)$. Taking the square-root on both sides of the above gives

$$T_E \leq Ch^r, \quad \text{and so, } |T_A^*| \leq Ch^r \overline{\|v\|}_1 \tag{5.22}$$

by (5.21). Similarly, we can show from Lemma 5.1 that

$$|T_B^*| = \left|\sum_{ij\in I_S}\{(h_{i+1}^2 - h_i^2)(u_{xy})_{i+\frac{1}{2},j+\frac{1}{2}}\}(v_{i,j+1} - v_{ij})\right| \leq Ch^2\overline{\|v\|}_1, \tag{5.23}$$

and

$$\begin{aligned} T_C^* &= \sum_{(i,j+1)\in\Gamma_1^*\wedge(i,j)\in\Gamma_1^*} k_j h_i^2 (u_{xy})_{i+\frac{1}{2},j+\frac{1}{2}} \frac{v_{i,j+1} - v_{i,j}}{k_j} \\ &= \sum_{(i,j+1)\in\Gamma_1^*\wedge(i,j)\in\Gamma_1^*} h_i^{\frac{3}{2}}\sqrt{k_j}(u_{xy})_{i+\frac{1}{2},j+\frac{1}{2}} \times \sqrt{h_i k_j}\frac{v_{i,j+1} - v_{i,j}}{k_j} \\ &\leq Ch^{\frac{3}{2}}\|u_{xy}\|_{0,\Gamma_1^*}\overline{\|v\|}_1. \end{aligned} \tag{5.24}$$

Combining (5.20) and (5.22)–(5.24) gives

$$\left|\sum_{ij\in I_S} h_i^2(u_{xy})_o(v_1 + v_4 - v_2 - v_3)\right| \leq C\{h^{\frac{3}{2}}\|\tilde{u}_{xy}\|_{0,\Gamma_1^*} + h^r + h^2\}\overline{\|v\|}_1. \tag{5.25}$$

The desired result (5.19) follows from $\|\tilde{u}_{xy}\|_{0,\Gamma_1^*} \leq C$ and Assumption A4. This completes the proof of Lemma 5.3. □

Lemma 5.4. *Let* A4–A6 *hold. For* $\sigma = \frac{1}{2} + \mu$ *and* $v = -\frac{3}{2} + \mu$, $\mu > 0$, *putting* $r = (p+1)\mu \leq 1.5$, *we have*

$$\left|\left(\widehat{\iint_{S_\Box}} - \iint_{S_\Box}\right)(u_I)_y v_y\right| \leq C\{h^r + h^3\|\tilde{u}_{xyyy}\|_{0,S} + T_0^*\} \times \overline{\|v\|}_1, \quad v \in V_h^0, \tag{5.26}$$

where

$$T_0^* = \sqrt{\sum_{ij\in I_S} \|h_i^3 \tilde{u}_{xxxx}\|_{0,\square_{ij}}^2} + h\sqrt{\sum_{ij\in I_S} \|h_i^2 \tilde{u}_{xxxy}\|_{0,\square_{ij}}^2}$$
$$+ h^2\sqrt{\sum_{ij\in I_S} \|h_i \tilde{u}_{xxyy}\|_{0,\square_{ij}}^2} + \sqrt{\sum_{ij\in I_S} \left\| \frac{h_i^4}{k_j} \tilde{u}_{yyyy}\right\|_{0,\square_{ij}}^2} \tag{5.27}$$

and $\tilde{u} = u(\xi,\eta)$ *for some* $(\xi,\eta) \in \square_{ij}$.

Proof. For the terms in (5.18), we have

$$\left|\sum_{ij\in I_S} \frac{h_i}{k_j} k_j^4(\tilde{u}_{yyyy})(v_1+v_4-v_2-v_3)\right| = \left|\sum_{ij\in I_S}(h_ik_j)k_j^3(\tilde{u}_{yyyy})\left(\frac{v_4-v_2}{k_j} - \frac{v_3-v_1}{k_j}\right)\right|$$
$$\le Ch^3\|\tilde{u}_{yyyy}\|_{0,S}^2 \overline{\|v\|}_1 \tag{5.28}$$

and

$$\left|\sum_{ij\in I_S} \frac{h_i}{k_j} h_i^3 k_j(\tilde{u}_{xxxy})(v_1+v_4-v_2-v_3)\right| \le \sum_{ij\in I_S} h_i^4 k_j|\tilde{u}_{xxxy}|\cdot\left(\left|\frac{v_4-v_2}{k_j}\right| + \left|\frac{v_3-v_1}{k_j}\right|\right)$$
$$\le Ch\sqrt{\sum_{ij\in I_S} \|h_i^2\tilde{u}_{xxxy}\|_{0,\square_{ij}}^2} \cdot \overline{\|v\|}_1. \tag{5.29}$$

Similarly, it can be shown that

$$\left|\sum_{ij\in I_S} \frac{h_i}{k_j} h_i^4(\tilde{u}_{xxxx})(v_1+v_4-v_2-v_3)\right| \le C\sqrt{\sum_{ij\in I_S}\left\|\frac{h_i^4}{k_j}\tilde{u}_{xxxx}\right\|_{0,\square_{ij}}^2} \cdot \overline{\|v\|}_1,$$
$$\left|\sum_{ij\in I_S} \frac{h_i}{k_j} h_i^2k_j^2(\tilde{u}_{xxyy})(v_1+v_4-v_2-v_3)\right| \le Ch^2\sqrt{\sum_{ij\in I_S}\|h_i\tilde{u}_{xxyy}\|_{0,\square_{ij}}^2} \cdot \overline{\|v\|}_1,$$
$$\left|\sum_{ij\in I_S} \frac{h_i}{k_j} h_ik_j^3(\tilde{u}_{xyyy})(v_1+v_4-v_2-v_3)\right| \le Ch^3\|\tilde{u}_{xyyy}\|_{0,S} \cdot \overline{\|v\|}_1. \tag{5.30}$$

From Lemma 5.3, (5.18), (5.28), (5.29), and (5.30), we obtain

$$\left|\left(\widehat{\iint_S} - \iint_S\right)(u_I)_y v_y\right|$$
$$\le C\{h^{1.5} + h^3\|\tilde{u}_{xyyy}\|_{0,S} + h^3\|\tilde{u}_{yyyy}\|_{0,S} + T_0^*\} \times \overline{\|v\|}_1, \tag{5.31}$$

where T_0^* is given in (5.27). This completes the proof of Lemma 5.4. $\square$

We comment that a distinction of the above analysis from that in [8] is that (5.26) was derived separately for different mesh densities in the y-direction. In contrast, the bound in [8] corresponding with (5.31) was derived on a uniform mesh. We have the following theorem.

Theorem 5.5. *Let* A4–A6 *hold, and assume that the exact solution, u, is four-times continuously differentiable on S, except the x-derivatives of u at $x = 0$. Let $\sigma = \frac{1}{2} + \mu$, $\mu > 0$, and $r = (p+1)\mu \leq 1.5$. Then, we have*

$$T = \sup_{v \in V_h^0} \frac{1}{\|v\|_1} \left| \left(\widehat{\iint_{S_\square}} - \iint_{S_\square} \right) \nabla u_I \nabla v \right| = O(h^r). \tag{5.32}$$

Proof. We have from Lemmas 5.2 and 5.4,

$$\begin{aligned} &\left| \left(\widehat{\iint_{S_\square}} - \iint_{S_\square} \right) \{ (u_I)_x v_x + (u_I)_y v_y \} \right| \\ &\quad \leq C\{ h^r + h^{1.5} + h^3 \|\tilde{u}_{xyyy}\|_{0,S} + \overline{T}_0^* \} \|v\|_1 \end{aligned} \tag{5.33}$$

for $v \in V_h^0$, where

$$\begin{aligned} \overline{T}_0^* = {} & \sqrt{\sum_{ij \in I_S} \|h_i^3 \tilde{u}_{xxxx}\|_{0,\square_{ij}}^2} + h \sqrt{\sum_{ij \in I_S} \|h_i^2 \tilde{u}_{xxxy}\|_{0,\square_{ij}}^2} + h^{1.5} \sqrt{\sum_{ij \in I_S} \|h_i \tilde{u}_{xxyy}\|_{0,\square_{ij}}^2} \\ & + \sqrt{\sum_{ij \in I_S} \left\| \frac{k_j^{2.5}}{h_i} \tilde{u}_{yyyy} \right\|_{0,\square_{ij}}^2} + \sqrt{\sum_{ij \in I_S} \left\| \frac{h_i^4}{k_j} \tilde{u}_{yyyy} \right\|_{0,\square_{ij}}^2}. \end{aligned} \tag{5.34}$$

Obviously, we have from $\sigma > \frac{1}{2}$,

$$h^r + h^{1.5} + h^3 \|\tilde{u}_{xyyy}\|_{0,S} \leq Ch^r. \tag{5.35}$$

It was proved in [8] that

$$\sqrt{\sum_{ij \in I_S} \|h_i^3 \tilde{u}_{xxxx}\|_{0,\square_{ij}}^2} + \sqrt{\sum_{ij \in I_S} \|h_i^2 \tilde{u}_{xxxy}\|_{0,\square_{ij}}^2} + h^2 \sqrt{\sum_{ij \in I_S} \|h_i \tilde{u}_{xxyy}\|_{0,\square_{ij}}^2} \leq Ch^{1.5}. \tag{5.36}$$

Let us now consider the bounds for the last two terms, denoted by $I^{1/2}$ and $II^{1/2}$, respectively, on the right-hand side of (5.34). Because $u_{yyyyy} \in C(S)$, we obtain

$$I := h^3 \sum_{ij \in I_S} \left\| \frac{k_j^{2.5}}{h_i} \tilde{u}_{yyyy} \right\|_{0,\square_{ij}}^2 = h^3 \sum_{ij \in I_S} \iint_{\square_{ij}} \frac{k_j^5}{h_i^2} \tilde{u}_{yyyyy}^2 \leq Ch^3 \sum_{ij \in I_S} \frac{k_j^6}{h_i}. \tag{5.37}$$

Because $\sum_{i=1}^{N} \frac{1}{h_i} \leq C \frac{1}{i^p h^{p+1}}$, we have $\sum_{i=1}^{N} \frac{1}{h_i} \leq C \frac{1}{h^2} \ln \frac{1}{h}$ for $p = 1$ and $\sum_{i=1}^{N} \frac{1}{h_i} \leq C \frac{1}{h^2}$ for $p \neq 1$. Then, we have

$$I \leq Ch^3 \left(\sum_{j=1}^{N} h_j^6 \right) \left(\sum_{i=1}^{N} \frac{1}{h_i} \right) \leq Ch^8 \frac{1}{h^2} \ln \frac{1}{h} \leq Ch^6 \ln \frac{1}{h}. \tag{5.38}$$

Moreover, we have

$$\begin{aligned} II &:= \sum_{ij \in I_S} \left\| \frac{h_i^4}{k_j} \tilde{u}_{xxxx} \right\|_{0,\Box_{ij}}^2 \leq \sum_{ij \in I_S} \iint_{\Box_{ij}} \frac{h_i^8}{k_j^2} \tilde{u}_{xxxx}^2 \\ &\leq \sum_{ij \in I_S} \frac{h_i^9}{k_j} \tilde{u}_{xxxx}^2 \leq C \sum_{ij \in I_S} \frac{h_i^2}{k_j} h_i^7 x_i^{2\sigma-8} \leq Ch^2 \left(\sum_{i=1}^{N} h_i^7 x_i^{2\sigma-8} \right) \left(\sum_{j=1}^{N} \frac{1}{k_j} \right). \end{aligned} \tag{5.39}$$

Next, for $r \leq 1.5$ we obtain

$$\begin{aligned} \sum_{i=1}^{N} h_i^7 x_i^{2\sigma-8} &\leq C \sum_{i=1}^{N} i^{(2\sigma-1)(p+1)-7} h^{(2\sigma-1)(p+1)} \\ &\leq C \sum_{i=1}^{N} i^{2r-7} h^{2r} \leq C \frac{\ln N}{N^3} h^{2r}. \end{aligned} \tag{5.40}$$

Because $\sum_{j=1}^{N} \frac{1}{k_j} \leq C \frac{1}{h^2} (\ln \frac{1}{h})$, we obtain from (5.39) and (5.40),

$$II \leq Ch^{3+2r} \left(\ln \frac{1}{h} \right)^2. \tag{5.41}$$

Combining (5.33)–(5.36), (5.38), and (5.41) gives the desired result (5.32). This completes the proof of Theorem 5.5. □

Remark 5.6. Suppose that Assumption A5 is replaced by A1. Theorem 5.5 can be proven similarly by the arguments given above. However, the proof based on A5 is simpler than that based on A1. In the next section, we will give numerical experiments in which the solution satisfies A1, and the numerical results are in perfect agreement with the theoretical results in Theorem 3.1.

6. NUMERICAL EXPERIMENTS

Consider the following boundary value problem:

$$-\Delta u = f, \quad \text{in } S, \tag{6.1}$$

$$u = 0, \quad \text{on } \Gamma_1 \cup \Gamma_2, \tag{6.2}$$

$$\frac{\partial u}{\partial n} = 0, \quad \text{on } \Gamma_3 \tag{6.3}$$

where S is the triangle defined by $S = \{(x, y), 0 < x < 1, (y < \sqrt{3}x) \wedge (y < \sqrt{3}(1 - x))\}$, $\Gamma_1 = \{(x, 0) : 0 < x < 1\}$, $\Gamma_2 = \{(x, 1 - \sqrt{3}x) : \frac{1}{2} < x < 1\}$, and $\Gamma_3 = \{(x, \sqrt{3}x) : 0 < x < \frac{1}{2}\}$. The exact solution is chosen to be

$$u_{\text{exact}}(x, y) = \left(y + \sqrt{3}x\right)^\sigma y^\sigma\left(\sqrt{3} - \left(y + \sqrt{3}x\right)\right)\left(\frac{\sqrt{3}}{2} - y\right). \tag{6.4}$$

When $\sigma = 1$ or 2, the solution is smooth. However, when $\sigma < 2$ and $\sigma \neq 1$, the derivatives of u are unbounded along Γ_1.

For $\sigma < 2$ and $\sigma \neq 1$, we may use the stretching function in A3 for y_i,

$$y_i = d_i = (ih)^{p+1} \times \frac{\sqrt{3}}{2}, \quad i = 0, 1, \ldots, n, \tag{6.5}$$

where $h = \frac{1}{n}$, and

$$x_i = \frac{1}{\sqrt{3}} y_i, \quad x_{2n+1-i} = 1 - x_i, \;\; i = 0, 1, \ldots, n. \tag{6.6}$$

We first consider the numerical solution of the problem defined by (6.1), (6.2), and the Dirichlet condition on Γ_3 given by $u|_{\Gamma_3} = u_{\text{exact}}|_{\Gamma_3}$. We also assume that $\sigma = 2$, and thus the the solution (6.4) is smooth. Let S be partitioned into a uniform mesh with $N_S + 1$ mesh nodes along the y direction. The nodes of the mesh along the x-direction is determined automatically by the intersections of mesh lines in the y-direction and the diagonals Γ_3 and Γ_2. The problem is solved on this mesh by the FDM described in the previous sections, and the computed errors and condition numbers are listed in Table 1. In these results $\boldsymbol{A}$ denotes the system matrix arising from the Shortley–Weller difference approximation (2.4), and the condition number is defined by

$$Con(\boldsymbol{A}) = \frac{\lambda_{\max}(\boldsymbol{A})}{\lambda_{\min}(\boldsymbol{A})}, \tag{6.7}$$

where $\lambda_{\max}(\boldsymbol{A})$ and $\lambda_{\min}(\boldsymbol{A})$ are the maximum and minimum eigenvalues of $\boldsymbol{A}$, respectively. From Table 1, we can easily see the following relations:

$$\overline{\|\epsilon\|}_0 = O(h^2), \quad \overline{\|\epsilon\|}_1 = O(h^{1\frac{7}{8}}), \quad Con(\mathrm{A}) = O(h^{-2}), \tag{6.8}$$

TABLE 1 Errors and condition numbers for the smooth problem with $\sigma = 2$ and $p_1 = 1$

N_S	4	6	8	12	16	24
$\overline{\|\epsilon\|}_0$	7.96(−4)	3.42(−4)	1.90(−4)	8.36(−5)	4.69(−5)	2.08(−5)
$\overline{\|\epsilon\|}_1$	8.99(−3)	4.09(−3)	2.35(−3)	1.08(−3)	6.22(−4)	2.84(−4)
$\max_{ij}\|\epsilon_{ij}\|$	2.96(−3)	1.36(−3)	7.41(−4)	3.33(−4)	1.89(−4)	8.40(−5)
$\max_{ij}\|(\epsilon_x)_{i+\frac{1}{2},j}\|$	2.56(−2)	1.48(−2)	9.21(−3)	4.56(−3)	2.70(−3)	1.27(−3)
$\max_{ij}\|(\epsilon_y)_{i,j+\frac{1}{2}}\|$	3.56(−2)	2.15(−2)	1.41(−2)	7.31(−3)	4.45(−3)	2.14(−3)
ϵ_1	1.07(−3)	5.43(−4)	2.74(−4)	9.61(−5)	4.38(−5)	1.40(−5)
ϵ_2	2.39(−3)	5.03(−4)	2.82(−4)	1.33(−4)	6.80(−5)	2.40(−5)
$Con(\boldsymbol{A})$	6.40	13.8	25.4	57.4	103	232

where $\epsilon = u - u_h$. Note that for $\overline{\|\epsilon\|}_1$, there exist a loss of orders compared with the optimal order $\overline{\|\epsilon\|}_1 = O(h^2)$ of convergence. This confirms the convergence order $\overline{\|\epsilon\|}_1 = O(h^{1.5})$ in the error analysis in [5, 7].

Next, we choose $\sigma = \frac{7}{6}$ and consider the Poisson equation with the Dirichlet and Neumann conditions in (6.1)–(6.3). Numerical results are presented in Tables 2–5 with $p_1 = p + 1 = 1, 2, 2.25, 2.5$, respectively. All numerical computation in Tables 1–6 is carried out in double precision. Because $p_1(\sigma - \frac{1}{2}) = \frac{2}{3}p_1 = r$, we expect from Theorem 3.1 that

$$\overline{\|\epsilon\|}_1 = O(h^{\frac{2}{3}}), \quad \text{for } p_1 = 1, \tag{6.9}$$

$$\overline{\|\epsilon\|}_1 = O(h^{\frac{4}{3}}), \quad \text{for } p_1 = 2, \tag{6.10}$$

$$\overline{\|\epsilon\|}_1 = O(h^{1.5}), \quad \text{for } p_1 = 2.25, \tag{6.11}$$

$$\overline{\|\epsilon\|}_1 = O(h^{1.5}), \quad \text{for } p_1 = 2.5. \tag{6.12}$$

From Table 4, we see that the asymptotic convergence rates for $p_1 = 2.25$ in the different norms are, respectively,

$$\overline{\|\epsilon\|}_0 = O(h^2), \quad \overline{\|\epsilon\|}_1 = O(h^{1\frac{3}{4}}), \quad Con(\mathrm{A}) = O(h^{-3.5}), \tag{6.13}$$

$$\max_{ij}|\epsilon_{ij}| = O(h^{1\frac{2}{3}}), \quad \max_{ij}|(\epsilon_x)_{i+\frac{1}{2},j}| = O(h^{\frac{3}{4}}), \tag{6.14}$$

$$\max_{ij}|(\epsilon_y)_{i,j+\frac{1}{2}}| = O(h^{\frac{3}{4}}), \tag{6.15}$$

$$\epsilon_1 = O(h^{2.75}), \quad \epsilon_2 = O(h^{2.75}), \tag{6.16}$$

where $\epsilon_x = u_x - (u_h)_x$, $\epsilon_y = u_y - (u_h)_y$, and $\epsilon_1 = \max_i\{|\epsilon_{i,1}|\}$, $\epsilon_2 = \max_i\{|\epsilon_{i,2}|\}$. The superconvergence rate $\overline{\|\epsilon\|}_1 = O(h^{1\frac{7}{8}})$ is consistent with that of Theorem 3.1.

TABLE 2 Errors and condition numbers for the singular problem with $\sigma = \frac{7}{6}$ and $p_1 = 1$

N_S	4	6	8	12	16	24
$\overline{\|\epsilon\|}_0$	3.49(−3)	1.83(−3)	1.20(−3)	6.87(−4)	4.71(−4)	2.83(−4)
$\overline{\|\epsilon\|}_1$	1.12(−2)	6.10(−3)	4.14(−3)	2.49(−3)	1.77(−3)	1.11(−3)
$\max_{ij}\|\epsilon_{ij}\|$	9.66(−3)	5.28(−3)	3.28(−3)	1.72(−3)	1.12(−3)	7.21(−4)
$\max_{ij}\|(\epsilon_x)_{i+\frac{1}{2},j}\|$	3.22(−2)	1.75(−2)	1.20(−2)	7.60(−3)	5.55(−3)	3.63(−3)
$\max_{ij}\|(\epsilon_y)_{i,j+\frac{1}{2}}\|$	2.34(−2)	1.33(−2)	9.26(−3)	5.40(−3)	4.15(−3)	2.72(−3)
ϵ_1	9.66(−3)	3.82(−3)	2.54(−3)	1.54(−3)	1.11(−3)	7.00(−4)
ϵ_2	9.30(−3)	5.28(−3)	2.95(−3)	1.56(−3)	1.12(−3)	7.21(−4)
$Con(\boldsymbol{A})$	13.5	28.3	50.9	119	213	482

TABLE 3 Errors and condition numbers for the singular problem with $\sigma = \frac{7}{6}$ and $p_1 = 2$

N_S	4	6	8	12	16	24
$\overline{\|\epsilon\|}_0$	4.52(−3)	1.49(−3)	7.64(−4)	3.27(−4)	1.82(−4)	8.04(−5)
$\overline{\|\epsilon\|}_1$	2.27(−2)	9.49(−3)	5.40(−3)	2.55(−3)	1.50(−3)	7.12(−4)
$\max_{ij}\|\epsilon_{ij}\|$	1.38(−2)	4.35(−3)	2.45(−3)	1.03(−3)	6.23(−4)	3.18(−4)
$\max_{ij}\|(\epsilon_x)_{i+\frac{1}{2},j}\|$	6.40(−2)	2.88(−2)	1.71(−2)	1.09(−2)	8.54(−3)	5.37(−3)
$\max_{ij}\|(\epsilon_y)_{i,j+\frac{1}{2}}\|$	3.74(−2)	2.49(−2)	1.93(−2)	1.27(−2)	8.93(−3)	5.14(−3)
ϵ_1	1.58(−3)	3.91(−4)	2.01(−4)	7.97(−5)	4.07(−5)	1.58(−5)
ϵ_2	4.12(−3)	7.78(−4)	3.70(−4)	1.05(−4)	5.43(−5)	2.14(−5)
$Con(\boldsymbol{A})$	9.40	25.6	53.1	185	432	1.47(3)

TABLE 4 Errors and condition numbers for the singular problem with $\sigma = \frac{7}{6}$ and $p_1 = 2.25$

N_S	4	6	8	12	16	24
$\overline{\|\epsilon\|}_0$	5.83(−3)	1.92(−3)	9.34(−4)	3.90(−4)	2.16(−4)	9.53(−5)
$\overline{\|\epsilon\|}_1$	2.82(−2)	1.19(−2)	6.65(−3)	3.12(−3)	1.84(−3)	8.76(−4)
$\max_{ij}\|\epsilon_{ij}\|$	1.70(−2)	6.26(−3)	2.97(−3)	1.25(−3)	6.93(−4)	3.90(−4)
$\max_{ij}\|(\epsilon_x)_{i+\frac{1}{2},j}\|$	7.24(−2)	3.76(−2)	1.93(−2)	1.16(−2)	9.47(−3)	6.19(−3)
$\max_{ij}\|(\epsilon_y)_{i,j+\frac{1}{2}}\|$	4.84(−2)	2.68(−2)	2.10(−2)	1.42(−2)	1.02(−2)	6.35(−3)
ϵ_1	1.23(−3)	4.53(−4)	2.29(−4)	8.16(−5)	3.87(−5)	1.35(−5)
ϵ_2	3.36(−3)	8.88(−4)	3.34(−4)	1.26(−4)	6.22(−5)	2.21(−5)
$Con(\boldsymbol{A})$	11.2	35.7	79.3	314	790	3.00(3)

TABLE 5 Errors and condition numbers for the singular problem with $\sigma = \frac{7}{6}$ and $p_1 = 2.5$

N_S	4	6	8	12	16	24
$\overline{\|\epsilon\|}_0$	7.05(−3)	2.49(−3)	1.16(−3)	4.66(−4)	2.58(−4)	1.13(−4)
$\overline{\|\epsilon\|}_1$	2.32(−2)	1.47(−2)	8.09(−3)	3.73(−3)	2.20(−3)	1.05(−3)
$\max_{ij}\|\epsilon_{ij}\|$	1.97(−2)	8.35(−3)	3.46(−3)	1.48(−3)	8.29(−4)	4.60(−4)
$\max_{ij}\|(\epsilon_x)_{i+\frac{1}{2},j}\|$	7.78(−2)	4.61(−2)	2.45(−2)	1.25(−2)	1.02(−2)	6.97(−3)
$\max_{ij}\|(\epsilon_y)_{i,j+\frac{1}{2}}\|$	5.81(−2)	2.88(−2)	2.26(−2)	1.55(−2)	1.14(−2)	9.13(−3)
ϵ_1	1.02(−3)	4.53(−4)	2.11(−4)	6.72(−5)	2.94(−5)	9.14(−6)
ϵ_2	2.69(−3)	8.24(−4)	3.54(−4)	1.23(−4)	5.59(−5)	1.77(−5)
$Con(\boldsymbol{A})$	14.0	51.9	123	546	147	6.22(3)

From Tables 2–5, we also observe that the ratios of $\overline{\|\epsilon\|}_1$ are given by

$$\frac{\overline{\|\epsilon\|}_1 \text{ at } N_S = 12}{\overline{\|\epsilon\|}_1 \text{ at } N_S = 24} = \frac{2.49(-3)}{1.11(-3)} = 2.24 = 2^{1.16} = O(h^{\frac{10}{9}}),$$
$$p_1 = 1, \text{ from Table 2},$$
$$\frac{\overline{\|\epsilon\|}_1 \text{ at } N_S = 12}{\overline{\|\epsilon\|}_1 \text{ at } N_S = 24} = \frac{2.55(-3)}{7.12(-4)} = 3.58 = 2^{1.84} = O(h^{1\frac{4}{5}}),$$
$$p_1 = 2, \text{ from Table 3},$$
$$\frac{\overline{\|\epsilon\|}_1 \text{ at } N_S = 12}{\overline{\|\epsilon\|}_1 \text{ at } N_S = 24} = \frac{3.12(-3)}{8.76(-4)} = 3.56 = 2^{1.83} = O(h^{1\frac{4}{5}}),$$
$$p_1 = 2.25, \text{ from Table 4},$$
$$\frac{\overline{\|\epsilon\|}_1 \text{ at } N_S = 12}{\overline{\|\epsilon\|}_1 \text{ at } N_S = 24} = \frac{3.73(-3)}{1.05(-3)} = 3.55 = 2^{1.82} = O(h^{1\frac{4}{5}}),$$
$$p_1 = 2.5, \text{ from Table 5},$$

which are all coincident with the predictions in (6.9)–(6.12). When $p_1 = 1$, the poor rates, $O(h^{\frac{2}{3}})$ in theory and $O(h^{\frac{10}{9}})$ in computation, display a necessity using the local refinements to improve accuracy of the numerical solutions. Obviously, the optimal value $p_1 = 2.25$ is the best choice for computation due to its better accuracy than those of others.

Finally, we choose $\sigma = 0.95$ and the optimal value $p_1 = 3.333333333$; numerical results are provided in Table 6, from which we can see

$$\overline{\|\epsilon\|}_0 = O(h^2), \quad \overline{\|\epsilon\|}_1 = O(h^{1.85}), \quad Con(\boldsymbol{A}) = O(h^{-4.5}),$$
$$\max_{ij}|\epsilon_{ij}| = O(h^2), \quad \max_{ij}\left|(\epsilon_x)_{i+\frac{1}{2},j}\right| = O(h),$$

TABLE 6 Errors and condition numbers for the singular problem with $\sigma = 0.95$ and $p_1 = 3.3333333$

N_S	4	6	8	12	16	24
$\overline{\|\epsilon\|}_0$	1.22(−2)	5.22(−3)	2.46(−3)	9.58(−4)	5.26(−4)	2.32(−4)
$\overline{\|\epsilon\|}_1$	5.33(−2)	2.56(−2)	1.39(−2)	6.16(−3)	3.59(−3)	1.69(−3)
$\max_{ij}\|\epsilon_{ij}\|$	3.02(−2)	1.49(−2)	6.86(−3)	2.72(−3)	1.47(−3)	6.69(−4)
$\max_{ij}\|(\epsilon_x)_{i+\frac{1}{2},j}\|$	9.76(−2)	6.54(−2)	3.81(−2)	1.78(−2)	1.20(−2)	8.58(−3)
$\max_{ij}\|(\epsilon_y)_{i,j+\frac{1}{2}}\|$	9.49(−2)	4.69(−2)	3.65(−2)	3.16(−2)	3.09(−2)	3.14(−2)
ϵ_1	1.53(−3)	3.23(−4)	1.19(−4)	3.13(−5)	1.24(−5)	3.39(−6)
ϵ_2	7.90(−3)	1.14(−3)	3.50(−4)	8.06(−5)	3.05(−5)	8.07(−6)
$Con(\boldsymbol{A})$	34.7	199	575	3.72(3)	1.25(4)	7.44(4)

$$\max_{ij} \left|(\epsilon_y)_{i,j+\frac{1}{2}}\right| = O(h^{-\delta}), \quad 0 < \delta \ll 1,$$

$$\epsilon_1 = O(h^3), \quad \epsilon_2 = O(h^3).$$

Although the derivatives $\max_{ij}|(\epsilon_y)_{i+\frac{1}{2},j}| = O(h^{-\delta})$ with respect to y are divergent for $\sigma < 1$, the errors $\|\epsilon\|_1 = O(h^{1.85})$. Such results are also in good agreement with that of Theorem 3.1. Note that the above computation provides, for the *first* time, significant numerical experiments for unbounded derivatives on *nonrectangular* boundary Γ of the Poisson equation by the Shortley–Weller difference approximation. Note that the numerical experiments in this section are carried out for mixed type of the Neumann and Dirichlet boundary conditions. To our best knowledge, this is the first time to give meaningful numerical experiments for unbounded derivatives on *nonrectangular* boundary Γ_D of Poisson's equation by the Shortley–Weller difference approximation.

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