WILEY InterScience®
DISCOVER SOMETHING GREAT

Radial Basis Collocation Method and Quasi-Newton Iteration for Nonlinear Elliptic Problems

H. Y. Hu,[1] **J. S. Chen**[2]

[1] *Department of Mathematics, Tunghai University, Taichung 407, Taiwan Republic of China*

[2] *Department of Civil and Environmental Engineering, University of Califormia, Los Angeles, California 90095-1593*

Received 9 April 2007; accepted 31 January 2007
Published online 31 October 2007 in Wiley InterScience (www.interscience.wiley.com).
DOI 10.1002/num.20303

This work presents a radial basis collocation method combined with the quasi-Newton iteration method for solving semilinear elliptic partial differential equations. The main result in this study is that there exists an exponential convergence rate in the radial basis collocation discretization and a superlinear convergence rate in the quasi-Newton iteration of the nonlinear partial differential equations. In this work, the numerical error associated with the employed quadrature rule is considered. It is shown that the errors in Sobolev norms for linear elliptic partial differential equations using radial basis collocation method are bounded by the truncation error of the RBF. The combined errors due to radial basis approximation, quadrature rules, and quasi-Newton and Newton iterations are also presented. This result can be extended to finite element or finite difference method combined with any iteration methods discussed in this work. The numerical example demonstrates a good agreement between numerical results and analytical predictions. The numerical results also show that although the convergence rate of order 1.62 of the quasi-Newton iteration scheme is slightly slower than rate of order 2 in the Newton iteration scheme, the former is more stable and less sensitive to the initial guess. Numer Methods Partial Differential Eq 24: 991–1017, 2008

Keywords: radial basis functions; collocation method; quasi-Newton iteration; superlinear convergence; meshfree; semilinear equations

I. INTRODUCTION

Nonlinear elliptic partial differential equations represent many scientific and engineering problems. Some literatures concerning the numerical approaches for nonlinear equations have been given, for example, finite difference method [1–3], finite element method [4–6], and boundary element method [7]. The mesh-based methods are most widely used in solving partial differential equations. However, the construction of an appropriate mesh in arbitrary geometry is a

Correspondence to: H. Y. Hu, Department of Mathematics, Tunghai University, Taichung 407, Taiwan Republic of China (e-mail: huhy@math.thu.edu.tw)

hard and time-consuming task. Consequently, meshfree methods have been introduced in the last decade, based on approximation methods that do not require a mesh topology, for example, kernel estimate and modified kernel estimate [8–10], moving least-squares [11–15], partition of unity [16–20], natural neighbor interpolation [21], radial basis functions [22, 23], and non-uniform rational B-splines [24]. The discrete equations of the meshfree methods can be constructed based on smoothed collocation [8], Galerkin method [9, 11, 12, 16, 18, 21], least-squares method [25], collocation of strong form [26], and stabilized collocation of Galerkin method [27].

The collocation method can be interpreted as an extension of the least-squares methods. We view it as the least-squares method with quadrature rules [28, 29]. The numerical solutions by this method exhibit high accuracy, nevertheless it accompanies instability due to the large condition number. Several approaches have been proposed to address this issue, for example, the preconditioning techniques can be used to improve the stability [30]. A concept based on effective condition number was proposed to represent a more reasonable upper bound of condition number [28].

Various admissible functions, such as Fourier series, Chebyshev polynomials, and Legendre polynomials, have been used as the admissible functions in solving partial differential equations with success. The radial basis functions have also been employed in the construction of approximations for solving linear elliptic partial differential equation under the framework of collocation method [22, 23, 31]. The convergence rates obtained by this method are typically beyond those obtained by the conventional finite element and finite difference methods. Analogous to other meshfree methods, in radial basis collocation method the source points of radial basis functions and collocation nodes can be scattered or clustered in a rather arbitrary fashion.

In this work, we propose a discretization using radial basis collocation with the quasi-Newton iteration scheme to solve a semilinear elliptic partial differential equation and perform convergence analysis of this approach. For the collocation method using radial basis functions as admissible functions, we name such a scheme the radial basis collocation method. In the past two decades, there have been many applications for the radial basis functions, such as surface fitting, turbulence analysis, neural network, and partial differential equations, in which the lastest one is of interest to us. For the application in solving partial differential equations, a large amount of the literatures focused on the implementation scheme. Very limited results were presented in application to nonlinear problems using the radial basis collocation method. No error analysis and convergence study for such a discretization scheme was established to date.

The objective of this study is to investigate how a radial basis collocation method can be used to solve the semilinear elliptic partial differential equation and whether the sequence solutions converge. We extend the ideas of previous works [28, 31] to the analysis of the semilinear elliptic equations.

The remaining part of this paper is organized as follows. An introduction to radial basis functions is given in Section 2. In Section 3, the convergence properties of radial basis collocation method and quasi-Newton iteration scheme are established. Section 4 presents the implementation scheme and numerical experiments to demonstrate the effectiveness of the methods proposed. Some conclusions are summarized in Section 5.

II. THE RADIAL BASIS FUNCTIONS

There have been much development of radial basis functions (RBF) in the past two decades. Such basis functions can be used as the interpolatory tools for smooth functions. The convergence of interpolants for a continuous function has been discussed, for example, Madych [32] and Yoon [33]. Some applications to partial differential equations (PDEs) have been given. Kansa [22, 23] presented a series of applications in computational fluid dynamics. Franke and Schaback [34] provided some theoretical foundation of RBF method for solving PDE. Hon and Mao [35] presented an application to Burgers' equation. Wendland [36] derived error estimates for the solution of smooth problems. Hon [37] developed a quasi-radial basis functions method for solving the Black–Scholes equation. Hu et al. [29] presented a weighted radial basis collocation method for elasticity problems.

A. Rationale of RBF

Several forms of RBF have been introduced. A few examples are: multiquadrics (MQ), Gaussian, and thin plate splines RBFs as shown in Eqs. (2.1–2.3), respectively.

$$g_i(\mathbf{x}) = \left(r_i^2 + c^2\right)^{n-\frac{3}{2}}, n = 1, 2, \ldots, \text{ (multiquadrics)} \tag{2.1}$$

$$g_i(\mathbf{x}) = \exp\left(-\frac{r_i^2}{c^2}\right), \text{ (Gaussian)} \tag{2.2}$$

$$g_i(\mathbf{x}) = \begin{cases} r_i^{2n} \ln r_i, & n = 1, 2, \ldots, \text{ in } R^2, \\ r_i^{2n-1}, & n = 1, 2, \ldots, \text{ in } R^3, \end{cases} \text{ (thin plate splines)} \tag{2.3}$$

where $\mathbf{x} = (x, y)$ denotes the Cartesian coordinate in R^2, and the radius is given by

$$r_i = \{(x - x_i)^2 + (y - y_i)^2\}^{\frac{1}{2}},$$

where (x_i, y_i) is called the *source point* of the RBF, denoted by $\mathbf{x}_i$. The constant c involved in forms (2.1) and (2.2) is called the *shape parameter* of the RBF. These two functions become much flatter as the parameter c increases. The MQ RBF is called reciprocal MQ RBF if $n = 1$, linear MQ RBF if $n = 2$, and cubic MQ RBF if $n = 3$, and so forth. Among them, the linear MQ RBF is most widely used.

We assume that $\Omega \subset R^2$ is a closed region with the boundary $\partial\Omega$. Let $\mathcal{X}_s$ be a set of m source points,

$$\mathcal{X}_s = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\} \subseteq \Omega \cup \partial\Omega.$$

For a smooth function $u(\mathbf{x})$, we may approximate it by

$$v(\mathbf{x}) = \sum_{i=1}^{m} a_i g_i(\mathbf{x}), \quad \mathbf{x}_i \in \mathcal{X}_s, \tag{2.4}$$

where $v(\mathbf{x})$ is the approximation of $u(\mathbf{x})$, $m > 1$, and a_i are expansion coefficients. The convergence rate is given by [32]

$$|u(\mathbf{x}) - v(\mathbf{x})| \approx O(\eta^{c/\delta}), \tag{2.5}$$

for which $u(\mathbf{x})$ is globally analytic or band-limited function, $0 < \eta < 1$ is a real number, c denotes a shape parameter of the RBF, and δ denotes a *radial distance* defined as

$$\delta := \delta(\Omega, \mathcal{X}_s) = \sup_{\mathbf{x}\in\Omega} \min_{\mathbf{x}_i\in\mathcal{X}_s} \{(x - x_i)^2 + (y - y_i)^2\}^{\frac{1}{2}}. \tag{2.6}$$

Equation (2.5) indicates that the parameters c and δ affect the rate of convergence.

B. Fitting and Other Applications

Let the RBF approximation expressed by (2.4) be used to solve partial differential equations. The coefficients a_i in (2.4) can be determined by solving a set of linear equations arise from the collocation method (CM). The basic idea of CM is to force the residuals to be zero at the collocation poiints. We choose the collocation points to be located both in the domain and on the boundary so that a discrete system can be constructed. Let $\mathcal{X}_c$ be a set of N collocation points,

$$\mathcal{X}_c = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N\} \subseteq \Omega \cup \partial\Omega.$$

The sets $\mathcal{X}_s$ and $\mathcal{X}_c$ may or may not have the common points. For the mathematical theory about such an unsysmmetic approach, we refer the readers to [31].

We express such an approximation of a function $u(\mathbf{x})$ as follows

$$(u - v)(\mathbf{x}_j) = 0, \quad \text{for all } \mathbf{x}_j \in \mathcal{X}_c. \tag{2.7}$$

The set of collocation points $\mathcal{X}_c$ can be arbitrary scattered. Equation (2.7) can be written in a general form as

$$\sum_{i=1}^{m} a_i g_i(\mathbf{x}_j) = u(\mathbf{x}_j), \quad \text{for all } \mathbf{x}_j \in \mathcal{X}_c. \tag{2.8}$$

For the number of collocation points, we typically choose it to be larger than the number of the source points. Equation (2.8) leads to an algebraic system in the form

$$\mathcal{G}\mathbf{a} = \mathbf{b}, \tag{2.9}$$

where $\mathbf{a}$ is an unknown vector consisting of coefficients a_i in (2.4). It is clear that $\mathcal{G}$ is a N by m matrix with entries $g_i(\mathbf{x}_j)$, and $\mathbf{b}$ is a vector associated with function $u(\mathbf{x}_j)$ with dimension N. The resultant linear system (2.9) is an overdetermined system, and methods such as QR decomposition, singular value decomposition, or least-squares method can be considered.

While the radial basis functions offer good approximation of global functions, applications such as in computational biophysics and computational biochemistry often exhibit local properties, and the employment of small shape parameter c in basis functions for the solution of these problems yields very acute results. The Poisson–Boltzmann equation [38, 39] is typical of this kind, where nonlinearity and singularity exist. One possible approach for solving Poisson–Boltzmann equation is to introduce radial basis functions with collocation method. This meshfree scheme is more adequate for such application as the complex affine map in meshing and the need of fine mesh in finite element [38,40,41] or finite difference [39] yield high computational costs.

III. RBF COLLOCATION METHOD WITH ITERATION SCHEME

Consider the semilinear elliptic partial differential equation of the form

$$\begin{aligned} -\Delta u &= f(\mathbf{x}, u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

where $\Omega \subset R^2$ is a bounded domain with boundary $\partial\Omega$. A numerical treatment for such a semilinear problem is as follows: Given an initial guess $u_0(\mathbf{x})$ and solve,

$$\begin{cases} -\Delta u_{n+1}(\mathbf{x}) = f(\mathbf{x}, u_n(\mathbf{x})), & \text{in } \Omega, \\ u_{n+1}(\mathbf{x}) = 0, & \text{on } \partial\Omega, \end{cases}$$

for $n = 0, 1, 2, \ldots$. For problems with strong nonlinearities, an alternative numerical treatment is introduced as follows: Given an initial guess $u_0(\mathbf{x})$ and solve,

$$\begin{cases} -\Delta u_{n+1}(\mathbf{x}) - \lambda u_{n+1}(\mathbf{x}) = f(\mathbf{x}, u_n(\mathbf{x})) - \lambda u_n(\mathbf{x}), & \text{in } \Omega, \\ u_{n+1}(\mathbf{x}) = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

for $n = 0, 1, 2, \ldots$. Such an iterative scheme falls into three types:

(1) the monotone iteration if λ is chosen to be a constant;
(2) the quasi-Newton iteration if λ is chosen to be a quotient, $\frac{f(\mathbf{x},u_n)-f(\mathbf{x},u_{n-1})}{u_n-u_{n-1}}$;
(3) the Newton iteration if λ is chosen to be a derivative, $f_u(\mathbf{x}, u_n) = \frac{\partial f(\mathbf{x},u_n)}{\partial u}$.

In this work, we consider the use of the radial basis collocation method to solve this semilinear problem.

A. Description of Radial Basis Collocation Method

Denote by $H^k(\Omega)$ the Sobolev space equipped with the norms $\|\cdot\|_{k,\Omega}, k = 0, 1, 2$. Let the solution of (3.1) be expressed as

$$u = \sum_{i=1}^{\infty} a_i g_i(\mathbf{x}), \tag{3.3}$$

where $g_i(\mathbf{x})$ are RBF, and a_i are the expansion coefficients. Such an expansion converges exponentially to the true solution u. The admissible functions can be chosen as

$$v = \sum_{i=1}^{L} \tilde{a}_i g_i(\mathbf{x}) \quad \text{in } \Omega, \tag{3.4}$$

where $\tilde{a}_i$ are unknown coefficients to be sought. Denote by V_L the finite dimensional collection of such admissible functions.

Choosing an initial u_0, a sequence of radial basis function iterates can be generated. The minimization problem is given as follow: To seek $u_{n+1} \in V_L$ such that

$$E(u_{n+1}) = \min_{v \in V_L} E(v), \tag{3.5}$$

where

$$E(v) = \frac{1}{2} \int_{\Omega} (\Delta v + \lambda v + f(\mathbf{x}, u_n) - \lambda u_n)^2 d\mathbf{x} + \frac{1}{2} \int_{\partial\Omega} v^2 d\ell, \tag{3.6}$$

for $u_n \in V_L, n = 0, 1, 2, \ldots$. The u_n in (3.5) is specified as

$$u_n := u_n(\mathbf{x}) = \sum_{i=1}^{L} a_i^{(n)} g_i(\mathbf{x}), \tag{3.7}$$

where $a_i^{(n)}$ denote the coefficients of the nth iterate u_n. The minimization problem (3.5) can be described equivalently the following formulation

$$B(u_{n+1}, v) = F(u_n, v), \quad \forall v \in V_L, \tag{3.8}$$

where

$$B(u, v) = \int_\Omega (\Delta u \Delta v + \lambda^2 uv)\, d\mathbf{x} + \int_{\partial\Omega} uv\, d\ell, \tag{3.9}$$

$$F(u, v) = \int_\Omega (\lambda^2 uv - f(\mathbf{x}, u)\Delta v)\, d\mathbf{x}. \tag{3.10}$$

The integrals in (3.6), (3.9), and (3.10) can be numerically evaluated using quadrature rules, for example, the Gaussian or the Newton–Cotes quadratures rules. The number of quadrature points in the domain and on the boundary are denoted by N_a and N_b, respectively. For the numbers $N_a, N_b \geq 1$,

$$\hat{\int_\Omega} P(\mathbf{x}) d\mathbf{x} := \sum_{k=1}^{N_a} w_k P(\xi_k), \quad \xi_k \in \Omega,$$

$$\hat{\int_{\partial\Omega}} P(\mathbf{x}) d\ell := \sum_{l=1}^{N_b} w_l P(\xi_l), \quad \xi_l \in \partial\Omega,$$

where $P(\mathbf{x})$ is a given function, w_k and w_l denote positive weights, and ξ_k and ξ_l denote the quadrature points located in the domain and on the boundary, respectively.

The collocation method can be interpreted as a least-squares method involving integration quadrature. The problem (3.5) involving quadrature approximation leads to a discrete problem as follows: To seek $\bar{u}_{n+1} \in V_L$ such that

$$\hat{E}(\bar{u}_{n+1}) = \min_{v \in V_L} \hat{E}(v), \tag{3.11}$$

where

$$\hat{E}(v) = \frac{1}{2} \hat{\int_\Omega} (\Delta v + \lambda v + f(\mathbf{x}, \bar{u}_n) - \lambda \bar{u}_n)^2 d\mathbf{x} + \frac{1}{2} \hat{\int_{\partial\Omega}} v^2 d\ell, \tag{3.12}$$

for $\bar{u}_n \in V_L, n = 0, 1, 2, \ldots$, in which $\bar{u}_n$ is expressed as

$$\bar{u}_n := \bar{u}_n(\mathbf{x}) = \sum_{i=1}^{L} \bar{a}_i^{(n)} g_i(\mathbf{x}), \tag{3.13}$$

where $\bar{a}_i^{(n)}$ denote the coefficients of the nth iterate $\bar{u}_n$.

Let N_c be the number of collocation points located in the domain and on the boundary, i.e., $N_c = N_a + N_b$. The discrete functional $\hat{E}(v)$ can further be manipulated as

$$\begin{aligned} \hat{E}(v) &= \frac{1}{2} \sum_{k=1}^{N_a} w_k \{(\Delta v + \lambda v + f(\mathbf{x}, \bar{u}_n) - \lambda \bar{u}_n)(\xi_k)\}^2 + \frac{1}{2} \sum_{l=1}^{N_b} w_l \{v(\xi_l)\}^2, \\ &=: \frac{1}{2} \mathbf{a}^T \mathbf{M}^T \mathbf{W} \mathbf{M} \mathbf{a} - \mathbf{a}^T \mathbf{M}^T \mathbf{W} \mathbf{b} + d, \end{aligned} \tag{3.14}$$

where the vector **a** consists of unknown coefficients with dimension L, and

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_1 \\ \mathbf{M}_2 \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} \mathbf{W}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}.$$

where $\mathbf{M}_1$ is an N_a by L matrix associated with $\Delta v + \lambda v$, $\mathbf{M}_2$ is an N_b by L matrix associated with v, and consequently $\mathbf{M}$ is an N_c by L matrix. The matrix $\mathbf{W}_1$ is an N_a by N_a diagonal weight matrix consisting of w_k, and $\mathbf{W}_2$ is an N_b by N_b diagonal weight matrix consisting of w_l. The vector $\mathbf{b}_1$ is the known vector associated with $f(\mathbf{x}, \bar{u}_n) - \lambda \bar{u}_n$, $\mathbf{b}_2$ is a zero vector, and d is the known value associated with $(f(\mathbf{x}, \bar{u}_n) - \lambda \bar{u}_n)^2$.

Therefore, the radial basis collocation method can be derived from such a discrete minimization problem. Taking variation with respect to the undetermined coefficient vector **a**, we obtain a linear system

$$\mathbf{M}^T \mathbf{W} \mathbf{M}\, \mathbf{a} = \mathbf{M}^T \mathbf{W}\, \mathbf{b}, \tag{3.15}$$

where $\mathbf{M}^T\mathbf{W}\mathbf{M}$ is symmetric and positive definite. The system (3.15) is called the normal equation. The direct collocation of (3.2) at quadrature points of $\hat{E}(v)$ yields

$$\mathbf{M}\mathbf{a} = \mathbf{b}. \tag{3.16}$$

Equation (3.16) can be solved by QR decomposition or singular value decomposition [42]. The solution of (3.15) approximates the solution of (3.16) bounded by a relative error shown as follows.

Proposition 3.1. *We assume that* $\mathbf{a}_s$ *is the solution of* (3.15) *and* $\mathbf{a}_o$ *is an optimal solution of* (3.16). *Then there exists a relative error*

$$\frac{\|\mathbf{a}_o - \mathbf{a}_s\|}{\|\mathbf{a}_s\|} \le \mathcal{E} \cdot \mathrm{Cond}(\mathbf{M}^T\mathbf{W}\mathbf{M}) \cdot \frac{\|\mathbf{M}^T\mathbf{W}\|}{\|\mathbf{M}^T\mathbf{W}\mathbf{b}\|},$$

where $\|\cdot\|$ *stands for the Euclidean norm, and* $\mathcal{E}$ *is an energy norm defined as*

$$\mathcal{E} = \|\mathbf{M}\mathbf{a}_o - \mathbf{b}\|,$$

and the $\mathrm{Cond}(\cdot)$ *denotes the condition number of a given matrix. The solution* $\mathbf{a}_s$ *approaches the solution* $\mathbf{a}_o$ *when the energy error* $\mathcal{E}$ *is very small*

The overdetermined system (3.16) can be written as follows

$$\{\Delta \bar{u}_{n+1} + \lambda \bar{u}_{n+1} + f(\mathbf{x}, \bar{u}_n) - \lambda \bar{u}_n\}(\xi_k) = 0, \quad \text{for all } \xi_k \in \Omega, \tag{3.17}$$

$$\bar{u}_{n+1}(\xi_l) = 0, \quad \text{for all } \xi_l \in \partial\Omega. \tag{3.18}$$

Such a form can also be constructed in a weighted residual formulation for problem (3.2) using Dirac delta function as the test function. In general, the collocation points, i.e., the quadrature points, can be scattered in a rather arbitrary fashion.

B. Convergence Analysis for General Scheme

In this section, we discuss the convergence properties of the sequence $\bar{u}_{n+1}$ of the problem (3.11) to the solution u. The minimization problems are equivalent to the variational formulations. The analysis made in this section based on variational formulations consists of two steps. The first one is for the original formulation without quadrature approximation, and the second one is for the discrete formulation where the quadrature rules are considered. We summarize the conditions necessary for our discussion in the following assumption.

Assumption 3.1. Let $u \in H^2(\Omega)$ be a general solution of the problem (3.1), where the function $f(\mathbf{x}, u)$ is nonlinear in u. There exists constants γ and μ such that

$$-\lambda_1 < \gamma \le \frac{f(\mathbf{x},u) - f(\mathbf{x},w)}{u - w} \le \mu \le \lambda, \quad \forall(\mathbf{x},u),(\mathbf{x},w) \in \bar{\Omega} \times B_\varepsilon, \tag{3.19}$$

where $\bar{\Omega} = \Omega \cup \partial\Omega$, and for sufficiently small $\varepsilon > 0$, and B_ε is defined as

$$B_\varepsilon = \{v \in H^2(\Omega) : \|u - v\|_0 \le \varepsilon + \|u - u_0\|_0\}, \tag{3.20}$$

where $\|\cdot\|_0 = \|\cdot\|_{0,\Omega}$, and $u_0 \in H^2(\Omega)$ is a given function. The sequence with any arbitrary initial guess in B_ε will converge to u in general. Here, λ is given in (3.2), $\lambda_1 > 0$ is the smallest eigenvalue of operator $-\Delta$ in Ω subject to the homogeneous Dirichlet boundary condition.

For the purpose of error analysis, we assume that there exists a projection $u_L \in V_L$ satisfying

$$B(u_L, v) = F(u, v), \quad \forall v \in V_L. \tag{3.21}$$

Combining with formulation (3.8) and using mathematical induction, we obtain an estimate in Sobolev zero norm $\|\cdot\|_{0,\Omega}$ (abbreviated as $\|\cdot\|_0$).

Lemma 3.1. *Let Assumption* 3.1 *hold, u be the exact solution of* (3.1) *and u_L be the solution of* (3.21). *If $u_0, u_1, \ldots, u_n$ all lie in B_ε, then there exists*

$$\|u_L - u_{n+1}\|_0 \le \alpha\, \|u - u_n\|_0, \tag{3.22}$$

where $0 < \alpha = \frac{\lambda^2 - \gamma\lambda_1}{\lambda^2 + \lambda_1^2} < 1$. *Moreover,*

$$\|u - u_{n+1}\|_0 \le \frac{1 - \alpha^{n+1}}{1 - \alpha}\|u - u_L\|_0 + \alpha^{n+1}\|u - u_0\|_0 \tag{3.23}$$

Proof. Subtracting (3.8) from (3.21), we have

$$B(u_L - u_{n+1}, v) = F(u - u_n, v), \quad \forall v \in V_L,$$

and

$$\int_\Omega \Delta(u_L - u_{n+1})\Delta v\, d\mathbf{x} + \int_\Omega \lambda^2(u_L - u_{n+1})v\, d\mathbf{x} + \int_{\partial\Omega}(u_L - u_{n+1})v\, d\ell$$
$$= \int_\Omega \lambda^2(u - u_n)v\, d\mathbf{x} - \int_\Omega (f(\mathbf{x},u) - f(\mathbf{x},u_n))\Delta v\, d\mathbf{x}.$$

Denote by $\varepsilon_{n+1} = u_L - u_{n+1}$ for all $n \geq 0$. Let $v = \varepsilon_{n+1}$, the above equation becomes

$$\int_\Omega \Delta\varepsilon_{n+1}\Delta\varepsilon_{n+1}\,d\mathbf{x} + \int_\Omega \lambda^2 \varepsilon_{n+1}\varepsilon_{n+1}\,d\mathbf{x} + \int_{\partial\Omega} \varepsilon_{n+1}\varepsilon_{n+1}\,d\ell$$
$$= \int_\Omega \lambda^2(u - u_n)\varepsilon_{n+1}\,d\mathbf{x} - \int_\Omega \frac{f(\mathbf{x},u) - f(\mathbf{x},u_n)}{u - u_n}(u - u_n)\Delta\varepsilon_{n+1}\,d\mathbf{x}.$$

It follows that

$$\begin{aligned}
&({\lambda_1}^2 + \lambda^2)\|\varepsilon_{n+1}\|_{0,\Omega}^2 + \|\varepsilon_{n+1}\|_{0,\partial\Omega}^2 \\
&\leq \lambda^2\|u - u_n\|_{0,\Omega}\|\varepsilon_{n+1}\|_{0,\Omega} - \gamma\|u - u_n\|_{0,\Omega}\|\Delta\varepsilon_{n+1}\|_{0,\Omega} \\
&\leq (\lambda^2 - \gamma\lambda_1)\|u - u_n\|_{0,\Omega}\|\varepsilon_{n+1}\|_{0,\Omega},
\end{aligned}$$

where a inequality is used

$${\lambda_1}^2\|\varepsilon_{n+1}\|_{0,\Omega}^2 \leq \|\Delta\varepsilon_{n+1}\|_{0,\Omega}^2.$$

Moreover, we obtain

$$({\lambda_1}^2 + \lambda^2)\|\varepsilon_{n+1}\|_{0,\Omega}^2 \leq (\lambda^2 - \gamma\lambda_1)\|u - u_n\|_{0,\Omega}\|\varepsilon_{n+1}\|_{0,\Omega}.$$

Consequently, we have an estimate as follows

$$\|u_L - u_{n+1}\|_0 = \|\varepsilon_{n+1}\|_0 \leq \frac{\lambda^2 - \gamma\lambda_1}{\lambda^2 + \lambda_1^2}\|u - u_n\|_0 =: \alpha\|u - u_n\|_0.$$

This is the first desired result (3.22).

The second part of the proof is by induction. Suppose that $u \neq u_{n+1}$ almost every where in solution domain Ω, we obtain

$$\begin{aligned}
&\|u - u_{n+1}\|_0 \leq \|u - u_L\|_0 + \|u_L - u_{n+1}\|_0 \\
&\leq \|u - u_L\|_0 + \alpha\|u - u_n\|_0 \\
&\leq \|u - u_L\|_0 + \alpha\|u - u_L\|_0 + \alpha\|u_L - u_n\|_0 \\
&= (1 + \alpha)\|u - u_L\|_0 + \alpha^2\|u - u_{n-1}\|_0 \\
&\leq (1 + \alpha)\|u - u_L\|_0 + \alpha^2\|u - u_L\|_0 + \alpha^2\|u_L - u_{n-1}\|_0 \\
&= (1 + \alpha + \alpha^2)\|u - u_L\|_0 + \alpha^3\|u - u_{n-2}\|_0 \\
&\quad\vdots \\
&\leq (1 + \alpha + \alpha^2 + \cdots + \alpha^n)\|u - u_L\|_0 + \alpha^{n+1}\|u - u_0\|_0 \\
&= \frac{1 - \alpha^{n+1}}{1 - \alpha}\|u - u_L\|_0 + \alpha^{n+1}\|u - u_0\|_0.
\end{aligned}$$

This is the second desired result (3.23). ∎

Next, a variational formulation involving the Newton–Cotes quadrature rules is taken into account. The discrete problem (3.11) can be described equivalently: To seek $\bar{u}_{n+1} \in V_L$ such that

$$\hat{B}(\bar{u}_{n+1}, v) = \hat{F}(\bar{u}_n, v), \quad \forall v \in V_L, \tag{3.24}$$

where $\hat{B}(\cdot,\cdot)$ and $\hat{F}(\cdot,\cdot)$ stand for the terms $B(\cdot,\cdot)$ and $F(\cdot,\cdot)$ evaluated by quadratures rules,

$$\hat{B}(u,v) = \hat{\int_\Omega} (\Delta u \Delta v + \lambda^2 uv) d\mathbf{x} + \hat{\int_{\partial\Omega}} uv\, d\ell,$$

$$\hat{F}(u,v) = \hat{\int_\Omega} (\lambda^2 uv - f(\mathbf{x},u)\Delta v) d\mathbf{x}.$$

According to [31], there exists positive constant C such that

$$\|v\|_{k,\Omega} \le C_1 L^k \|v\|_{0,\Omega}, \quad \forall v \in V_L, \tag{3.25}$$

$$\|v\|_{k,\partial\Omega} \le C_2 L^k \|v\|_{0,\partial\Omega} \quad \forall v \in V_L, \tag{3.26}$$

where L denotes the number of admissible functions in (3.4), and C_i is a generic constant.

Lemma 3.2. *For the the Newton–Cotes quadrature rules with accuracy of order r, there exist the following bounds,*

$$\left\| \left(\int_\Omega - \hat{\int_\Omega} \right) (\Delta v)^2 \right\|_{0,\Omega} \le C\hbar^{r+1} L^{r+5} \|v\|^2_{0,\Omega}, \tag{3.27}$$

$$\left\| \left(\int_\Omega - \hat{\int_\Omega} \right) v^2 \right\|_{0,\Omega} \le C\hbar^{r+1} L^{r+1} \|v\|^2_{0,\Omega}, \tag{3.28}$$

$$\left\| \left(\int_{\partial\Omega} - \hat{\int_{\partial\Omega}} \right) v^2 \right\|_{0,\partial\Omega} \le C\hbar^{r+1} L^{r+1} \|v\|^2_{0,\partial\Omega}, \tag{3.29}$$

where $\hbar$ denotes the maximal spacing between integration nodes, i.e., collocation points

Proof. For the integration rules with accuracy of of order r, we have

$$\hat{\int_\Omega} P = \int_\Omega \hat{P},$$

where $\hat{P}$ is the interpolant polynomial of order r in Ω with the maximal spacing $\hbar$,

$$\hbar = \max_i \hbar_i,$$

where $\hbar_i$ is the maximal distance of integration node ξ_i of its neighbors. Then, we obtain

$$\left| \left(\int_\Omega - \hat{\int_\Omega} \right) P \right| = \left| \int_\Omega P - \int_\Omega \hat{P} \right| = \left| \int_\Omega (P - \hat{P}) \right| \le C\hbar^{r+1} \left| \int_\Omega P^{(r+1)} \right|,$$

where $P^{(r+1)}$ denotes the $r+1$th derivative of P. Let $P=(\Delta v)^2$, we have

$$\begin{aligned}
\left|\int_\Omega P^{(r+1)}\right| &= \left|\int_\Omega ((\Delta v)^2)^{(r+1)}\right| = \left|\int_\Omega \sum_{i=0}^{r+1} C_i^{r+1}(\Delta v)^{(r+1-i)}(\Delta v)^{(i)}\right| \\
&= \left|\sum_{i=0}^{r+1} C_i^{r+1}\int_\Omega (\Delta v)^{(r+1-i)}(\Delta v)^{(i)}\right| \le C\sum_{i=0}^{r+1}\left|\int_\Omega (\Delta v)^{(r+1-i)}(\Delta v)^{(i)}\right| \\
&\le C\sum_{i=0}^{r+1}|\Delta v|_{r+1-i,\Omega}|\Delta v|_{i,\Omega} \le C\sum_{i=0}^{r+1}|v|_{r+3-i,\Omega}|v|_{i+2,\Omega} \\
&\le C\sum_{i=0}^{r+1} L^{r+3-i}\|v\|_{0,\Omega}L^{i+2}\|v\|_{0,\Omega} \le CL^{r+5}\|v\|_{0,\Omega}^2,
\end{aligned}$$

where C_i^{r+1} denotes binomial coefficient. Combining above two inequalities gives the desired result (3.27). Again, we obtain

$$\begin{aligned}
\left|\int_\Omega (v^2)^{(r+1)}\right| &= \left|\int_\Omega \sum_{i=0}^{r+1} C_i^{r+1} v^{(r+1-i)}v^{(i)}\right| \le C\sum_{i=0}^{r+1}|v|_{r+1-i,\Omega}|v|_{i,\Omega} \\
&\le C\sum_{i=0}^{r+1} L^{r+1-i}\|v\|_{0,\Omega}\,L^i\|v\|_{0,\Omega} \le CL^{r+1}\|v\|_{0,\Omega}^2.
\end{aligned}$$

The desired result (3.28) is then obtained. Equation (3.29) can be derived similarly. ■

The projection u_L satisfies $B(u_L,v)=F(u,v)$, whereas $\hat{B}(u_L,v)\neq\hat{F}(u,v)$. By using the inequalities (3.27)–(3.29), we obtain a relation

$$|\hat{B}(u_L,v)-\hat{F}(u,v)|\le T(\hbar,L),$$

where

$$\begin{aligned}
T(\hbar,L) &= C_2\hbar^{r+1}L^{r+5}\|u_L\|_{0,\Omega}\|v\|_{0,\Omega} + C_3\lambda^2\hbar^{r+1}L^{r+1}(\|u_L\|_{0,\Omega}+\|u\|_{0,\Omega})\|v\|_{0,\Omega} \\
&\quad + C_4\hbar^{r+1}L^{r+3}\|f\|_{0,\Omega}\|v\|_{0,\Omega} + C_5\hbar^{r+1}L^{r+1}\|u_L\|_{0,\partial\Omega}\|v\|_{0,\partial\Omega},
\end{aligned}$$

and C_i are generic constants. Moreover, we obtain

$$|\hat{B}(u_L,v)-\hat{F}(u,v)|\le C'\hbar^{r+1}L^{r+5}\|v\|_{0,\Omega},$$

where

$$C'=\max\{C_2\|u_L\|_{0,\Omega}, C_3\lambda^2(\|u_L\|_{0,\Omega}+\|u\|_{0,\Omega}), C_4\|f\|_{0,\Omega}, C_5\|u_L\|_{0,\partial\Omega}\}.$$

Assume there exists another projection, denoted by $\bar{u}_L$, which satisfies discrete variational formulation

$$\hat{B}(\bar{u}_L,v)=\hat{F}(u,v). \tag{3.30}$$

Note that the projection u_L is a limit of the sequence $\{u_n\}$, and $\bar{u}_L$ is a limit of the sequence $\{\bar{u}_n\}$. For the latter projcetion, a crucial condition

$$\hbar^{r+1}L^{r+5} = o(\hbar^k), \quad k \geq 0, \tag{3.31}$$

should be fulfilled.

Lemma 3.3. *Let conditions* (3.30) *and* (3.31) *hold, following the results in* [28,31]*, there exist positive constants $\mathcal{C}_k, k = 0, 1, 2$ independent of c, δ and L such that*

$$\|u - \bar{u}_L\|_k \leq \mathcal{C}_k(\eta_k{}^{c/\delta}) \approx \mathcal{O}\left(\eta_k{}^{c\sqrt{L}}\right),$$

where $0 < \eta_k < 1, k = 0, 1, 2$ is real number, and $\eta_k = \exp(-\theta_k)$ with $\theta_k > 0$

Remark 3.1. We conclude from Lemma 3.3 that the errors in Sobolev norms for linear elliptic partial differential equations using radial basis collocation method are bounded by the truncation error of the RBF.

Theorem 3.1. *Let u be the exact solution of* (3.1) *and $\bar{u}_L$ be the solution of* (3.30). *If $\bar{u}_0 = u_0$, and $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n$ all lie in B_ε, there exist the following condition*

$$\|u - \bar{u}_{n+1}\|_0 \leq \frac{1 - \alpha^{n+1}}{1 - \alpha}\|u - \bar{u}_L\|_0 + \alpha^{n+1}\|u - u_0\|_0, \tag{3.32}$$

where α is defined in Lemma 3.1

Proof. Subtracting (3.24) from (3.30), we have

$$\hat{B}(\bar{u}_L - \bar{u}_{n+1}, v) = F(u - \bar{u}_n, v), \quad \forall v \in V_L.$$

Denote by $\bar{\varepsilon}_{n+1} = \bar{u}_L - \bar{u}_{n+1}$ for all $n \geq 0$. Let $v = \bar{\varepsilon}_{n+1}$, the above equation becomes

$$\begin{aligned} &\int_\Omega \Delta\bar{\varepsilon}_{n+1}\Delta\bar{\varepsilon}_{n+1}d\mathbf{x} + \int_\Omega \lambda^2\bar{\varepsilon}_{n+1}\bar{\varepsilon}_{n+1}d\mathbf{x} + \int_{\partial\Omega} \bar{\varepsilon}_{n+1}\bar{\varepsilon}_{n+1}d\ell \\ &= \int_\Omega \lambda^2(u - \bar{u}_n)\bar{\varepsilon}_{n+1}\, d\mathbf{x} - \int_\Omega \frac{f(\mathbf{x}, u) - f(\mathbf{x}, \bar{u}_n)}{u - \bar{u}_n}(u - \bar{u}_n)\Delta\bar{\varepsilon}_{n+1}d\mathbf{x}. \end{aligned}$$

It follows that

$$(\lambda_1{}^2 + \lambda^2)\|\bar{\varepsilon}_{n+1}\|_{0,\Omega}^2 \leq (\lambda^2 - \gamma\lambda_1)\|u - \bar{u}_n\|_{0,\Omega}\ \|\bar{\varepsilon}_{n+1}\|_{0,\Omega},$$

and

$$\|\bar{u}_L - \bar{u}_{n+1}\|_0 = \|\bar{\varepsilon}_{n+1}\|_0 \leq \alpha\|u - \bar{u}_n\|_0,$$

where α is defined in Lemma 3.1. Suppose that $u \neq \bar{u}_{n+1}$ almost every where in solution domain Ω. The argument similar to the one used in Lemma 3.1 shows that

$$\|u - \bar{u}_{n+1}\|_0 \leq \frac{1 - \alpha^{n+1}}{1 - \alpha}\|u - \bar{u}_L\|_0 + \alpha^{n+1}\|u - u_0\|_0.$$

This proves (3.32). ∎

Remark 3.2. Theorem 3.1 implies that $\bar{u}_{n+1}$ remains in the ball B_ε, so that the first term on the right hand side of Eq. (3.32) is less than or equal to ε.

Corollary 3.1. *Based on Lemma 3.3 and Theorem 3.1, for sufficiently large L, there exists*

$$\|u - \bar{u}_{n+1}\|_0 \le C_2\left(\eta_0^{c\sqrt{L}}\right) + \alpha^{n+1}\|u - u_0\|_0,$$

where $C_2 = \frac{\mathcal{C}_0(1-\alpha^{n+1})}{1-\alpha}$ *is a constant independent of c and L, and* $\mathcal{C}_0$ *is given in Lemma* 3.3

Remark 3.3. Corollary 3.1 implies that the sequence solution $\bar{u}_{n+1}$ for semilinear elliptic partial differential equations using radial basis collocation iteration scheme converges exponentially to u with an exponential convergence rate with respect to c and L.

C. Superlinear Convergence for Quasi-Newton's Iteration

In the beginning of this section, we provided three iteration schemes for solving a semilinear partial differential equation. They converge to the exact solution with different convergence rates. For the first iteration scheme, the iterates converge monotonically to the solution. The rate of convergence is slow and cannot be determined in general. For better performance, the quasi-Newton and Newton iteration schemes can be adopted to accelerate the convergence. Their iterates converge superlinearly to the solution. The latter two schemes are closely related, and their conditions for convergence are almost the same.

Newton's iteration converges more rapidly than quasi-Newton's iteration in general. On the other hand, Newton's iteration is generally more expensive per iteration. Besides, Newton's iteration is not guaranteed to converge due to its sensitivity to the initial guess. In present work the quasi-Newton iteration scheme is considered. Its rate of convergence is superlinear, of order 1.62. Here, we provide an estimate in Sobolev zero norm. For further discussion, one more assumption is needed.

Assumption 3.2. Assume that there exist constants γ_1 and γ_2 such that

$$\gamma_1 \le f_{uu}(\mathbf{x}, u) \le \gamma_2, \quad \forall(\mathbf{x}, u) \in \bar{\Omega} \times B_\varepsilon,$$

where f_{uu} denotes the second partial derivative of f with respect to u, and B_ε is defined in Assumption 3.1.

Let $G(u) = \Delta u - f(\mathbf{x}, u)$, we assume that the projection u_L satisfies the condition

$$\int_\Omega G(u_L)v\, d\mathbf{x} = 0, \quad \forall v \in V_L, \tag{3.33}$$

and its divide difference form is given by

$$\int_\Omega \left\{G[u_n] + G[u_{n-1}, u_n](u_L - u_n) + \frac{1}{2}G_{uu}(\xi_n)(u_L - u_{n-1})(u_L - u_n)\right\} v d\mathbf{x} = 0. \tag{3.34}$$

Here $G_{uu}(\cdot)$ denotes the second derivative of $G(\cdot)$ with respect to u, ξ_n is a function belong to B_ε, and the divide differences are given by

$$G[a] := G(a), \quad G[a, b] := \frac{G(b) - G(a)}{b - a}.$$

The corresponding quasi-Newton's iteration scheme is

$$\int_{\Omega}\{G[u_n]+G[u_{n-1},u_n](u_{n+1}-u_n)\}v\,d\mathbf{x}=0. \tag{3.35}$$

Subtracting (3.34) from (3.35) and combining definition of $G[\cdot,\cdot]$, we obtain

$$\int_{\Omega}G[u_{n-1},u_n](u_{n+1}-u_L)v\,d\mathbf{x}=\frac{1}{2}\int_{\Omega}G_{uu}(\xi_n)(u_L-u_{n-1})(u_L-u_n)v\,d\mathbf{x}$$

and considering that the test function v equals zero on boundary

$$\begin{aligned}&\int_{\Omega}\frac{G(u_n)-G(u_{n-1})}{u_n-u_{n-1}}(u_{n+1}-u_L)v\,d\mathbf{x}\\ &=-\frac{1}{2}\int_{\Omega}\frac{\partial}{\partial u}\nabla((u_L-u_{n-1})(u_L-u_n))\nabla v\,d\mathbf{x}-\frac{1}{2}\int_{\Omega}f_{uu}(\xi_n)(u_L-u_{n-1})(u_L-u_n)v d\mathbf{x}.\end{aligned}$$

Consequently, we have

$$\begin{aligned}&\int_{\Omega}\frac{-\Delta(u_n-u_{n-1})}{u_n-u_{n-1}}(u_{n+1}-u_L)v d\mathbf{x}+\int_{\Omega}\frac{f(\mathbf{x},u_n)-f(\mathbf{x},u_{n-1})}{u_n-u_{n-1}}(u_{n+1}-u_L)v\,d\mathbf{x}\\ &=\frac{1}{2}\frac{\partial}{\partial u}\int_{\Omega}\nabla((u_L-u_{n-1})(u_L-u_n))\nabla v\,d\mathbf{x}+\frac{1}{2}\int_{\Omega}f_{uu}(\xi_n)(u_L-u_{n-1})(u_L-u_n)v\,d\mathbf{x},\end{aligned}$$

and it follows that

$$\begin{aligned}&(\lambda_1+\gamma)\|u_{n+1}-u_L\|_0\|v\|_0\\ &\le\frac{1}{2}\{C\|u_L-u_{n-1}\|_1\|u_L-u_n\|_1\|v\|_1+\gamma_2\|u_L-u_{n-1}\|_0\|u_L-u_n\|_0\|v\|_0\}\\ &\le\frac{1}{2}\{CL^3\|u_L-u_{n-1}\|_0\|u_L-u_n\|_0\|v\|_0+\gamma_2\|u_L-u_{n-1}\|_0\|u_L-u_n\|_0\|v\|_0\},\end{aligned}$$

in which two inequalities are used,

$$\lambda_1\,u_n\le-\Delta u_n,\quad \text{for all } n\ge 1,$$
$$\|v\|_k\le CL^k\|v\|_0,\ \ k\ge 1,\quad \text{for all } v\in V_L,$$

where C is a generic constant. According to the above, we have the estimate

$$\|u_L-u_{n+1}\|_0=\|u_{n+1}-u_L\|_0\le\frac{CL^3+\gamma_2}{2(\lambda_1+\gamma)}\|u_L-u_{n-1}\|_0\|u_L-u_n\|_0.$$

We write down above estimate as a lemma below.

Lemma 3.4. *Let Assumptions* 3.1 *and* 3.2 *hold and u_L be the solution of* (3.21). *If u_0 and u_1 are sufficiently close to u_L, then the solution u_{n+1} of formulation* (3.5) *converges to u_L. There exists an error bound*

$$\|u_L-u_{n+1}\|_0\le\tau\|u_L-u_{n-1}\|_0\|u_L-u_n\|_0, \tag{3.36}$$

where $\tau=\frac{CL^3+\gamma_2}{2(\lambda_1+\gamma)}>0$ is a constant independent of n

An estimate for formulation involving quadrature rules can be derived similarly. We have one more lemma as follows.

Lemma 3.5. *Let Lemma* 3.4 *and the condition* (3.31) *hold, and let* $\bar{u}_L$ *be solution of* (3.30). *If* $\bar{u}_0$ *and* $\bar{u}_1$ *are sufficiently close to* $\bar{u}_L$, *then the solution* $\bar{u}_{n+1}$ *of formulation* (3.24) *converges to* $\bar{u}_L$. *There exists an error bound*

$$\|\bar{u}_L - \bar{u}_{n+1}\|_0 \le \tau \ \|\bar{u}_L - \bar{u}_{n-1}\|_0 \|\bar{u}_L - \bar{u}_n\|_0, \tag{3.37}$$

where τ *is defined in Lemma 3.4*

Denote by $\bar{\varepsilon}_k$ the error $\bar{u}_L - \bar{u}_k$ for all $k \ge 0$. Consider the following situation

$$\|\bar{\varepsilon}_{n+1}\|_0 = \tau \|\bar{\varepsilon}_{n-1}\|_0 \|\bar{\varepsilon}_n\|_0. \tag{3.38}$$

To determine the order of convergence, we assume that there exists an asymptotic relationship as follow

$$\|\bar{\varepsilon}_{n+1}\|_0 = \omega \ \|\bar{\varepsilon}_n\|_0{}^q.$$

Therefore, it is clear that

$$\|\bar{\varepsilon}_n\|_0 = \omega \|\bar{\varepsilon}_{n-1}\|_0{}^q \quad \text{and} \quad \|\bar{\varepsilon}_{n-1}\|_0 = \omega^{-\frac{1}{q}} \|\bar{\varepsilon}_n\|_0{}^{\frac{1}{q}}.$$

Substituting the above equations into Eq. (3.38) leads to

$$\omega \|\bar{\varepsilon}_n\|_0{}^q = \tau \omega^{-\frac{1}{q}} \|\bar{\varepsilon}_n\|_0{}^{\frac{1}{q}} \|\bar{\varepsilon}_n\|_0$$

and

$$\omega^{1+\frac{1}{q}} \|\bar{\varepsilon}_n\|_0{}^q = \tau \|\bar{\varepsilon}_n\|_0{}^{1+\frac{1}{q}}.$$

We conclude that $q = 1 + \frac{1}{q}$, or $q = \frac{1+\sqrt{5}}{2} \approx 1.62$, and $\tau = \omega^q$. Hence, the quasi-Newton's rate of convergence is superlinear. Equation (3.38) can be written as

$$\|\bar{\varepsilon}_{n+1}\|_0 = \tau^{\frac{1}{q}} \|\bar{\varepsilon}_n\|_0{}^{1.62}.$$

Above results can be summarize as a theorem below.

Theorem 3.2. *Let the conditations in Lemma 3.5 hold. There exists a constant* ω *such that*

$$\|\bar{u}_L - \bar{u}_{n+1}\|_0 \le \omega \|\bar{u}_L - \bar{u}_n\|_0{}^q, \tag{3.39}$$

where $q \approx 1.62$ *and* $\omega = \tau^{\frac{1}{q}} = \tau^{q-1}$ *is a constant independent of n*

In computation, such a q-rate in the numerical results can be calculated via the formula

$$q = \frac{\ln \|\bar{\varepsilon}_{n+1}\|_S - \ln \|\bar{\varepsilon}_n\|_S}{\ln \|\bar{\varepsilon}_n\|_S - \ln \|\bar{\varepsilon}_{n-1}\|_S}, \tag{3.40}$$

where $\|\cdot\|_S$ stands for one kind of the Sobolev norms.

The Newton and quasi-Newton schemes are closely related. Recall the Taylor expansion of (3.33)

$$\int_\Omega \left\{ G(u_n) + G(u_n)(u_L - u_n) + \frac{1}{2}\, G_{uu}(\zeta_n)(u_L - u_n)^2 \right\} v\, d\mathbf{x} = 0, \tag{3.41}$$

where ζ_n is a function belonging to B_ε. The corresponding Newton scheme is

$$\int_\Omega \{G(u_n) + G(u_n)(u_{n+1} - u_n)\} v\, d\mathbf{x} = 0. \tag{3.42}$$

After similar derivations as those for quasi-Newton method, for Newton method there exists an estimate

$$\|\bar{u}_L - \bar{u}_{n+1}\|_0 \le \tau \|\bar{u}_L - \bar{u}_n\|_0{}^2, \tag{3.43}$$

where τ is defined in Lemma 3.4.

In conclusion, let $\{\bar{u}_n^N\}$ and $\{\bar{u}_n^Q\}$ be generated by Newton and quasi-Newton schemes approaching a limit $\bar{u}_L$, respectively. Suppose that the initial guesses are quite close to the desired solution $\bar{u}_L$. There exist the following estimates

$$\begin{aligned}
&\left\|\bar{u}_L - \bar{u}_n^N\right\|_0 \le \tau \left\|\bar{u}_L - \bar{u}_{n-1}^N\right\|_0^2, \\
&\left\|\bar{u}_L - \bar{u}_n^Q\right\|_0 \le \omega \left\|\bar{u}_L - \bar{u}_{n-1}^Q\right\|_0^q, \quad q \approx 1.62.
\end{aligned}$$

It follows that

$$\begin{aligned}
\left\|\bar{u}_L - \bar{u}_n^N\right\|_0 &\le \tau^{1+2+\cdots+2^{n-1}} \{\|\bar{u}_L - u_0\|_0\}^{2^n} \\
&= \tau^{2^n-1} \{\|\bar{u}_L - u_0\|_0\}^{2^n} = \frac{1}{\tau} \{\tau \|\bar{u}_L - u_0\|_0\}^{2^n}, \\
\left\|\bar{u}_L - \bar{u}_n^Q\right\|_0 &\le \omega^{1+q+\cdots+q^{n-1}} \{\|\bar{u}_L - u_0\|_0\}^{q^n} \\
&= \omega^{\frac{q^n-1}{q-1}} \{\|\bar{u}_L - u_0\|_0\}^{q^n} = \frac{1}{\tau} \{\tau \|\bar{u}_L - u_0\|_0\}^{q^n}.
\end{aligned}$$

To satisfy a desired tolerance ϵ, we must have

$$n \ge \frac{\kappa}{\ln 2}, \quad \text{for the RBC-N iteration scheme,} \tag{3.44}$$

$$n \ge \frac{\kappa}{\ln q}, \quad \text{for the RBC-QN iteration scheme,} \tag{3.45}$$

where κ is given by

$$\kappa = \ln \left\{ \frac{\ln \tau + \ln \epsilon}{\ln \tau + \ln \|\bar{u}_L - u_0\|_0} \right\}.$$

Numerical experiments concerning the superlinear convergence will be given in the next section.

IV. NUMERICAL EXPERIMENTS

In this section, we provide an algorithm for the radial basis collocation method combined with the quasi-Newton iteration and present an example in a two-dimensional setting.

A. Implementation Schemes

The quasi-Newton iteration scheme for the semilinear problem (3.1) is given by

$$\begin{aligned} -\Delta\bar{u}_{n+1}(\mathbf{x}_k) - \frac{f(\mathbf{x},\bar{u}_n(\mathbf{x}_k)) - f(\mathbf{x},\bar{u}_{n-1}(\mathbf{x}_k))}{\bar{u}_n(\mathbf{x}_k) - \bar{u}_{n-1}(\mathbf{x}_k)}\bar{u}_{n+1}(\mathbf{x}_k) \\ = f(\mathbf{x}_k,\bar{u}_n(\mathbf{x}_k) - \frac{f(\mathbf{x},\bar{u}_n(\mathbf{x}_k)) - f(\mathbf{x},\bar{u}_{n-1}(\mathbf{x}_k))}{\bar{u}_n(\mathbf{x}_k) - \bar{u}_{n-1}(\mathbf{x}_k)}\bar{u}_n(\mathbf{x}_k), \quad \mathbf{x}_k \in \Omega, \\ \bar{u}_{n+1}(\mathbf{x}_l) = 0, \quad \mathbf{x}_l \in \partial\Omega, \end{aligned} \tag{4.1}$$

for $n = 1, 2, \ldots$, with initial guesses $\bar{u}_0(\mathbf{x})$ and $\bar{u}_1(\mathbf{x})$, in which $\bar{u}_{n+1}(\mathbf{x})$ is approximated as

$$\bar{u}_{n+1}(\mathbf{x}) = \sum_{i=1}^{L} \bar{a}_i^{(n+1)} g_i(\mathbf{x}) = \Phi(\mathbf{x})^T \mathbf{a}^{(n+1)}, \tag{4.2}$$

where

$$\Phi(\mathbf{x}) = [g_1(\mathbf{x}), g_2(\mathbf{x}), \ldots, g_L(\mathbf{x})]^T, \quad \mathbf{a}^{(n+1)} = \left[\bar{a}_1^{(n+1)}, \bar{a}_2^{(n+1)}, \ldots, \bar{a}_L^{(n+1)}\right]^T,$$

and $g_i(\mathbf{x})$ are radial basis functions mentioned in Section 2.

Let $\{\mathbf{x}_k\}_{k=1}^{N_a}$ and $\{\mathbf{x}_l\}_{l=1}^{N_b}$ be the sets of collocation points located in the domain and on the boundary, respectively. Denote by $N_c = N_a + N_b$ the total number of collocation points. The analysis in the previous section states that the sequence $\bar{u}_{n+1}$ will converge to solution u as long as the condition (3.31) is fulfilled. This condition requires that the maximal spacing of collocation points satisfy the relation

$$\hbar = o\left(L^{-\frac{r+5}{r+1-k}}\right) = o\left(L^{-\left(1+\frac{4+k}{r+1-k}\right)}\right), \quad k \geq 0.$$

To ensure that an optimal solution can be obtained, the number of collocation points should be chosen much larger than the number of basis functions, i.e., $N_c \geq L$.

The algebraic linear system (4.1) can be further written as

$$\mathbf{M}\mathbf{a}^{(n+1)} - \mathbf{Q}(\mathbf{a}^{(n-1)}, \mathbf{a}^{(n)})\mathbf{a}^{(n+1)} = \mathbf{b}(\mathbf{a}^{(n)}) - \mathbf{Q}(\mathbf{a}^{(n-1)}, \mathbf{a}^{(n)})\mathbf{a}^{(n)}, \tag{4.3}$$

with two initial guesses $\mathbf{a}^{(0)}$ and $\mathbf{a}^{(1)}$, where $\mathbf{M}$ and $\mathbf{Q}(\cdot,\cdot)$ are N_c by L matrices, and the vector $\mathbf{b}(\cdot)$ is with dimension N_c. The definition of each row for the matrix $\mathbf{M}$ is

$$\mathbf{M}_j = \begin{cases} \Phi_{xx}(\mathbf{x}_j)^T + \Phi_{yy}(\mathbf{x}_j)^T, & 1 \leq j \leq N_a \\ \Phi(\mathbf{x}_j)^T, & N_a + 1 \leq j \leq N_c \end{cases},$$

TABLE I. The error norms and condition numbers for $c = 1.6$ and $N_c = 13 \times 25$.

	$L = N^2$			
	6^2	8^2	10^2	12^2
$\|u - \bar{u}_n\|_{\infty,\Omega}$	4.48 (−3)	1.53 (−4)	1.31 (−5)	1.74 (−6)
$\|u - \bar{u}_n\|_{0,\Omega}$	1.17 (−3)	5.80 (−5)	6.87 (−6)	8.93 (−7)
$\|u - \bar{u}_n\|_{1,\Omega}$	9.47 (−3)	4.94 (−4)	3.10 (−5)	2.94 (−6)
Cond.	4.24 (5)	1.11 (8)	8.55 (8)	2.63 (9)
No. of iterations	15	16	16	17

TABLE II. The error norms and condition numbers for $L = 10^2$ and $c = 1.6$.

	N_c			
	9×17	13×25	17×33	21×41
$\|u - \bar{u}_n\|_{\infty,\Omega}$	5.27 (−4)	1.31 (−5)	1.71 (−5)	2.63 (−5)
$\|u - \bar{u}_n\|_{0,\Omega}$	2.77 (−4)	6.87 (−6)	4.71 (−6)	6.43 (−6)
$\|u - \bar{u}_n\|_{1,\Omega}$	8.18 (−4)	3.10 (−5)	4.34 (−5)	7.70 (−5)
Cond.	8.54 (8)	8.55 (8)	2.14 (9)	4.37 (9)
No. of iterations	16	16	16	16

and for the matrix $\mathbf{Q}(\cdot,\cdot)$ is

$$\mathbf{Q}(\mathbf{a}^{(n-1)}, \mathbf{a}^{(n)})_j = \begin{cases} \frac{f(\mathbf{x}_j, \Phi(\mathbf{x}_j)^T \mathbf{a}^{(n)}) - f(\mathbf{x}_j, \Phi(\mathbf{x}_j)^T \mathbf{a}^{(n-1)})}{\Phi(\mathbf{x}_j)^T \mathbf{a}^{(n)} - \Phi(\mathbf{x}_j)^T \mathbf{a}^{(n-1)}} \Phi(\mathbf{x}_j)^T, & 1 \le j \le N_a, \\ \mathbf{0}, & N_a + 1 \le j \le N_c. \end{cases}$$

The components of the vector $\mathbf{b}(\cdot)$ are given by

$$\mathbf{b}(\mathbf{a}^{(n)})_j = \begin{cases} f(\mathbf{x}_j, \Phi(\mathbf{x}_j)^T \mathbf{a}^{(n)}), & 1 \le j \le N_a \\ 0, & N_a + 1 \le j \le N_c \end{cases}.$$

We name the iteration scheme (4.3) the radial basis collocation quasi-Newton (RBC-QN) iteration scheme. The stopping criterion for the iteration is chosen as

$$\|\bar{u}_{n+1}(\mathbf{x}) - \bar{u}_n(\mathbf{x})\|_{\infty,\Omega} \le 10^{-6}. \tag{4.4}$$

In fact, the Newton and the quasi-Newton iteration schemes are closely related. The radial basis collocation Newton (RBC-N) iteration scheme is given by

$$\mathbf{M}\mathbf{a}^{(n+1)} - \mathbf{J}(\mathbf{a}^{(n)})\mathbf{a}^{(n+1)} = \mathbf{b}(\mathbf{a}^{(n)}) - \mathbf{J}(\mathbf{a}^{(n)})\mathbf{a}^{(n)}, \tag{4.5}$$

with an initial guess $\mathbf{a}^{(0)}$. The term $\mathbf{J}(\cdot)$ is the Jacobian matrix of the vector $\mathbf{b}(\cdot)$, which is defined as

$$\mathbf{J}(\mathbf{a}^{(n)})_j = \begin{cases} f_u(\mathbf{x}_j, \Phi(\mathbf{x}_j)^T \mathbf{a}^{(n)}) \Phi(\mathbf{x}_j)^T, & 1 \le j \le N_a, \\ \mathbf{0}, & N_a + 1 \le j \le N_c. \end{cases}$$

Let the following approximation be used in the Newton iteration scheme (4.5).

$$f_u(\mathbf{x}, \bar{u}_n) \doteq \frac{f(\mathbf{x}, \bar{u}_n) - f(\mathbf{x}, \bar{u}_{n-1})}{\bar{u}_n - \bar{u}_{n-1}}$$

TABLE III. The error norms and condition numbers for $L = 10^2$ and $N_c = 325$.

	c				
	1.0	1.4	1.8	2.2	2.6
$\|u - \bar{u}_n\|_{\infty,\Omega}$	9.99 (−5)	1.41 (−5)	1.14 (−5)	8.54 (−6)	6.42 (−6)
$\|u - \bar{u}_n\|_{0,\Omega}$	2.47 (−5)	7.24 (−6)	6.09 (−6)	4.25 (−6)	2.70 (−6)
$\|u - \bar{u}_n\|_{1,\Omega}$	4.03 (−4)	2.32 (−5)	3.39 (−5)	2.77 (−5)	1.90 (−5)
Cond.	8.40 (8)	1.17 (9)	9.70 (8)	1.43 (9)	1.05 (9)
No. of iterations	16	16	16	16	16

TABLE IV. The iteration errors and q-rate for $L = 10^2$, $c = 1.6$, and $N_c = 325$.

	nth iteration					
	$n = 11$	$n = 12$	$n = 13$	$n = 14$	$n = 15$	$n = 16$
$\|\bar{u}_L - \bar{u}_n\|_{\infty,\Omega}$	2.93 (−1)	6.19 (−2)	5.22 (−3)	9.56 (−5)	1.45 (−7)	6.49 (−12)
q-rate	–	2.27	1.89	1.76	1.70	1.64
$\|\bar{u}_L - \bar{u}_n\|_{0,\Omega}$	2.23 (−1)	4.43 (−2)	3.61 (−3)	6.46 (−5)	9.69 (−8)	3.52 (−12)
q-rate	–	2.08	1.80	1.72	1.67	1.63
$\|\bar{u}_L - \bar{u}_n\|_{1,\Omega}$	5.47 (−1)	1.08 (−1)	8.80 (−3)	1.58 (−4)	2.37 (−7)	1.84 (−11)
q-rate	–	–	2.13	1.85	1.74	1.62

We then obtain a quasi-Newton iteration scheme (4.3). Note that the Newton iteration requires $2 \times N_a$ function evaluations per iteration for $f(\mathbf{x}, \bar{u}_n)$ and $f_u(\mathbf{x}, \bar{u}_n)$, whereas the quasi-Newton requires only N_a function evaluations per iteration for $f(\mathbf{x}, \bar{u}_n)$. Therefore, the Newton scheme is generally more expensive per iteration. Another disadvantage of Newton iteration is related to the sensitivity to the initial guess; it is not guaranteed to converge.

B. Numerical Test

Consider a two-dimensional model problem [1] of the form

$$\begin{aligned} -\Delta u &= f(x, y, u) \quad \text{in} \quad \Omega, \\ u_x &= 0 \ \text{ on } \ x = 0, \quad 0 \le y \le 2, \\ u &= 0 \ \text{ on } \ y = 0, \quad 0 < x \le 1, \\ u &= 0 \ \text{ on } \ x = 1, \quad 0 \le y \le 2, \\ u &= 0 \ \text{ on } \ y = 2, \quad 0 \le x \le 1, \end{aligned}$$

where $\Omega = \{(x, y) | 0 < x < 1, 0 < y < 2\}$, the function $f(x, y, u)$ is given by

$$f(x, y, u) = \frac{\pi^2}{4} u(1 - u) + q(x, y),$$

in which the function $q(x, y) = 2\sin(\frac{\pi y}{2}) + \frac{\pi^2}{4}(1 - x^2)^2 \sin^2(\frac{\pi y}{2})$. The analytical solution is given by

$$u(x, y) = (1 - x^2) \sin\left(\frac{\pi y}{2}\right).$$

TABLE V. The total errors for $L = 10^2$, $c = 1.6$, and $N_c = 325$.

	nth iteration						
	$n = 11$	$n = 12$	$n = 13$	$n = 14$	$n = 15$	$n = 16$	$n = 17$
$\|u - \bar{u}_n\|_{\infty,\Omega}$	2.39 (−1)	6.19 (−2)	5.22 (−3)	8.32 (−5)	1.30 (−5)	1.31 (−5)	1.31 (−5)
$\|u - \bar{u}_n\|_{0,\Omega}$	2.23 (−1)	4.43 (−2)	3.61 (−3)	5.81 (−5)	6.78 (−6)	6.87 (−6)	6.87 (−6)
$\|u - \bar{u}_n\|_{1,\Omega}$	5.47 (−1)	1.08 (−1)	8.80 (−3)	1.43 (−4)	3.09 (−5)	3.10 (−5)	3.10 (−5)

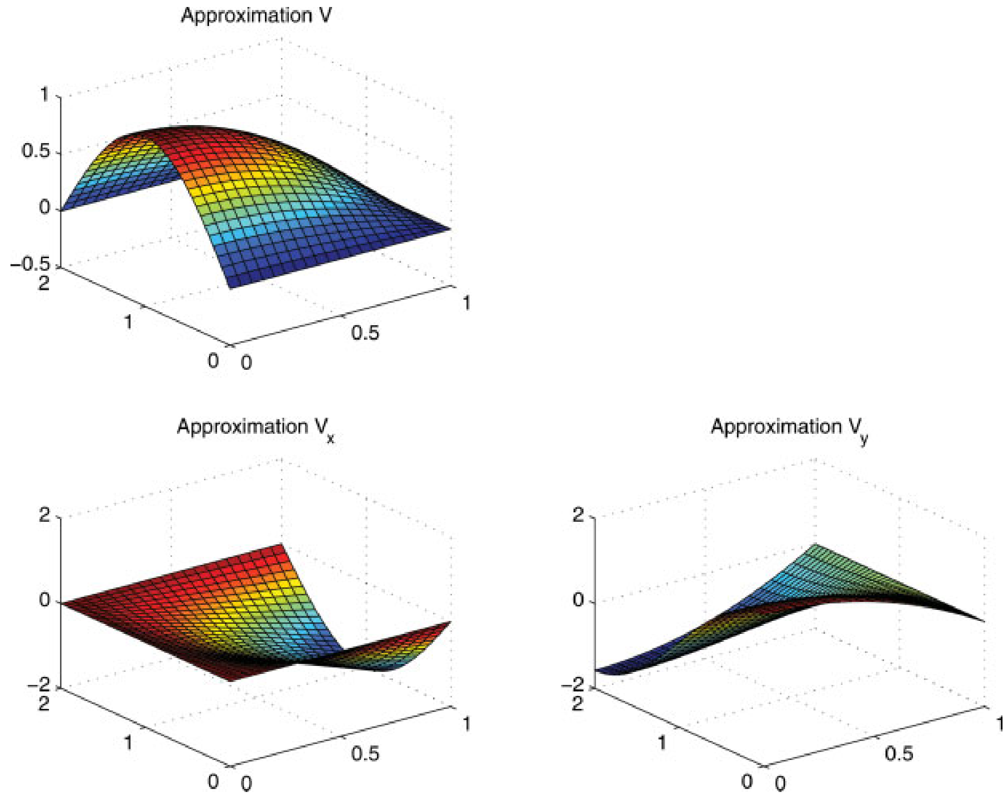

FIG. 1. The profiles of solution and solution derivatives by RBC-QN iteration scheme with $L = 11^2$, $N_c = 325$, and $c = 1.6$. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

We utilize the RBC-QN iteration scheme (4.3) with two initial guesses $\mathbf{a}^{(0)} = [-1, -1, \ldots, -1]^T$ and $\mathbf{a}^{(1)} = [1, 1, \ldots, 1]^T$ and use the following basis function

$$g_i(\mathbf{x}) = \frac{1}{\sqrt{r_i^2 + c^2}},$$

which is known as the reciprocal MQ RBF. Numerical results are obtained by taking $L = N^2$ to be 6^2–12^2, N_c to be 150–900, and c to be 1.0–3.0. Equally spaced collocation and source points are adopted so that the radial distance δ has an approximate relation as

$$\delta \approx O\left(\frac{1}{N}\right) \approx O\left(\frac{1}{\sqrt{L}}\right).$$

The number of collocation points, N_c, should be greater than the number of basis functions, L. In the analysis, an overdetermined algebraic system is constructed, the least-squares method is used to solve the algebraic system at each step, and the iteration is performed until criterion (4.4) is satisfied.

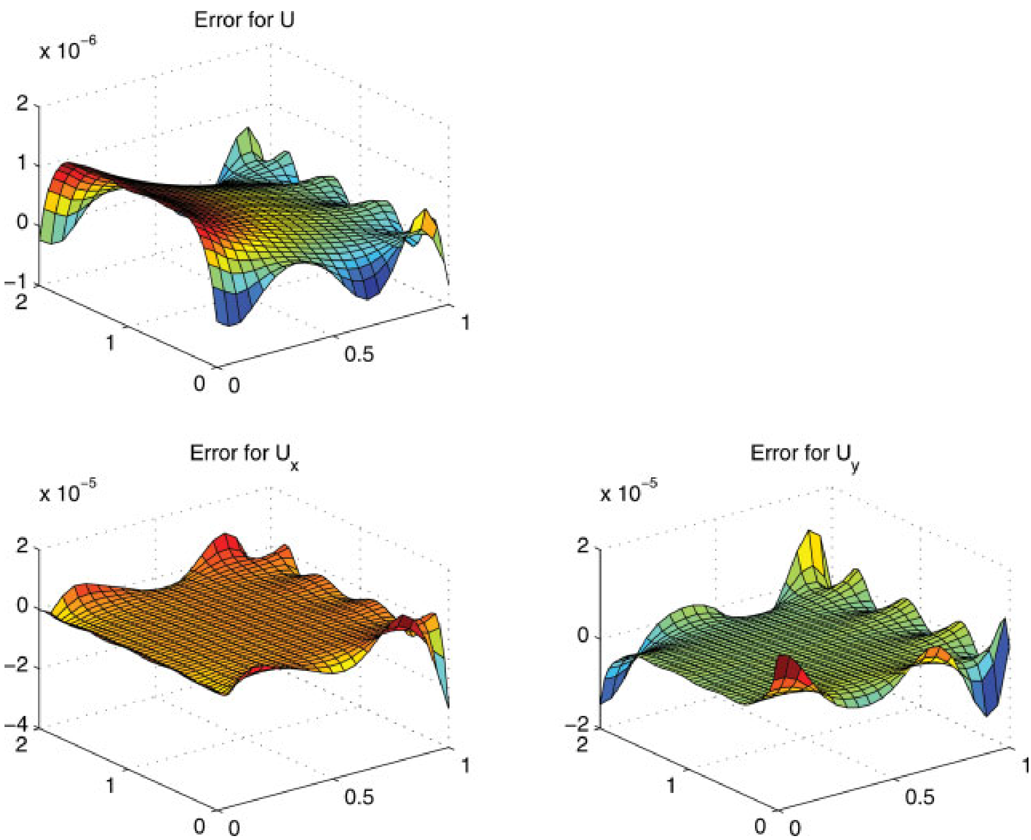

FIG. 2. The errors of solution and solution derivatives by RBC-QN iteration scheme with $L = 11^2$, $N_c = 325$, and $c = 1.6$. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

The computed errors in the nth iterate and condition number of the matrix $\mathbf{M} - \mathbf{Q}(\mathbf{a}^{(n-1)}, \mathbf{a}^{(n)})$ with various values of L are listed in Table I. We see from this table that there exist the following asymptotic properties

$$\begin{aligned}
\|u - \bar{u}_n\|_{\infty,\Omega} &= O\big((0.27)^{1/\delta}\big) = O\big((0.27)^{\sqrt{L}}\big),\\
\|u - \bar{u}_n\|_{0,\Omega} &= O\big((0.30)^{1/\delta}\big) = O\big((0.30)^{\sqrt{L}}\big),\\
\|u - \bar{u}_n\|_{1,\Omega} &= O\big((0.26)^{1/\delta}\big) = O\big((0.26)^{\sqrt{L}}\big).\\
\text{Cond.} &= O\big((4.28)^{1/\delta}\big) = O\big((4.28)^{\sqrt{L}}\big).
\end{aligned} \tag{4.6}$$

Above relations indicate that the errors decay exponentially as L increases, i.e., δ decreases, while the condition numbers grow exponentially as L increases.

Next, we consider the effects of the number of collocation points N_c and the shape parameter c of RBF. The computed results are listed in Tables II and III, respectively. We notice form Table II that the number of collocation points N_c has minor effect on the accuracy, as long as

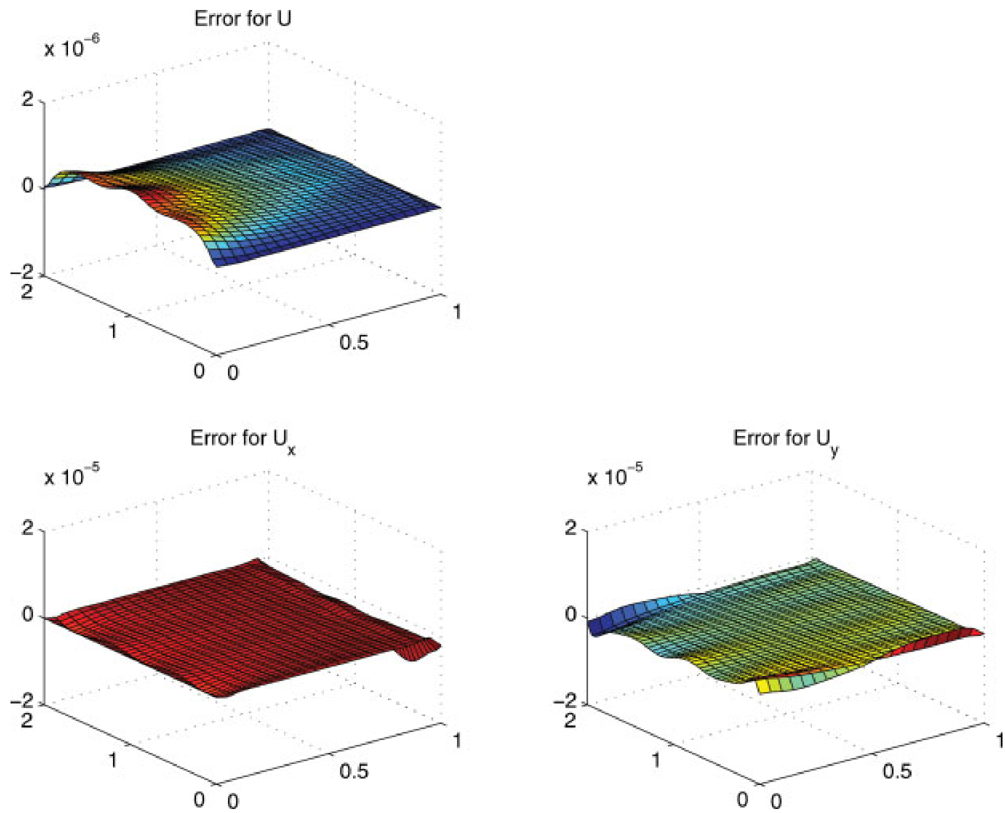

FIG. 3. The errors of solution and solution derivatives by penalty RBC-QN iteration scheme with penalty number $P_c = 10^6$. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

$N_c \geq 13 \times 25 = 325$. From Table III, we obtain the asymptotic relations as follows

$$\begin{aligned} \|u - \bar{u}_n\|_{\infty,\Omega} &= O((0.18)^c), \\ \|u - \bar{u}_n\|_{0,\Omega} &= O((0.25)^c), \\ \|u - \bar{u}_n\|_{1,\Omega} &= O((0.15)^c), \end{aligned} \tag{4.7}$$

The errors decay exponentially as c increases. Theoretical analysis states that the flatter RBF used for interpolation, the better accuracy in the approximation is obtained.

It is seen from above computational results that the number of RBF and its shape have major effects on the accuracy. The solution convergence can be achieved either by adding the basis functions or by increasing the shape parameter. Equations (4.6) and (4.7) demonstrate the exponential convergence, which agrees with the analysis given in Theorem 3.1 and Corollary 3.1.

The results presented so far are focused on the total errors. Next, we investigate the iteration errors. The 20th iterate is chosen to approximate the $\bar{u}_L$ and the iteration errors for $L = 10^2$ are provided in Table IV. By virtue of the formula (3.40), the q-rates for the iteration errors are calculated. We observe form Table IV that there exists a superlinear reduction. The predicted q-rates agree well with the convergence analysis result given in Theorem 3.2.

TABLE VI. The iteration errors and the quadratic rate for $L = 10^2$, $c = 1.6$, and $N_c = 325$ by using the Newton iteration scheme.

	nth iteration				
	$n = 7$	$n = 8$	$n = 9$	$n = 10$	$n = 11$
$\|\bar{u}_L - \bar{u}_n\|_{\infty,\Omega}$	1.23	3.03 (−1)	2.50 (−2)	1.82 (−4)	9.39 (−9)
q-rate	–	–	3.09	2.33	2.15
$\|\bar{u}_L - \bar{u}_n\|_{0,\Omega}$	1.03	2.25 (−1)	1.73 (−2)	1.22 (−4)	6.23 (−9)
q-rate	–	–	2.72	2.22	2.10
$\|\bar{u}_L - \bar{u}_n\|_{1,\Omega}$	2.55	5.49 (−1)	4.21 (−2)	2.98 (−4)	1.52 (−8)
q-rate	–	–	5.28	2.56	2.22

The total errors for the case of $L = 10^2$ are listed in Table V. The errors contain discretization errors and iteration errors. The profiles of approximate solutions and the errors for the case of $L = 11^2$ are plotted in Figs. 1 and 2, respectively. We can see from Fig. 2 that there seems to have larger errors occurred on the boundary. This can be improved by considering higher weight on the boundary in functionals $E(v)$ and $\hat{E}(v)$ defined in (3.5) and (3.11), for example,

$$E(v) = \frac{1}{2}\int_{\Omega}(\Delta v + \lambda v + f(\mathbf{x}, u_n) - \lambda u_n)^2 d\mathbf{x} + \frac{P_c}{2}\int_{\partial\Omega} v^2 d\ell,$$

where P_c denotes a large and positive number. The profiles of the errors by such an approach with $P_c = 10^6$ are illustrated in Fig. 3. Indeed, we observe that the errors near boundary have been improved. Readers may refer the related paper [29] for more details.

Besides, we demonstrate the results by using Newton scheme (4.5) as well. Taking the initial guess $\mathbf{a}^{(0)} = [1, 1, \ldots, 1]^T$, the iteration errors and the total errors for the case of $L = 10^2$ are given in Tables VI and VII, respectively. We observe from Table VI that there exists a quadratic error reduction and it needs fewer iterations to achieve the desired accuracy. More detailed observation of changes of total error in different norms are illustrated in Figure 4.

The radial basis function approximation can be applied to nonuniform point discretization. In this study, the random point distribution is generated by taking perturbation of point locations in the discretization with uniform point distribution by $\mathbf{x}_i' = \mathbf{x}_i + \beta_i \Delta\mathbf{x}$, where $\mathbf{x}_i$ is the position vector of point i of the uniform discretization, $\Delta\mathbf{x} = [\Delta x_1, \Delta x_2]$ is the nodal spacing vector of the uniform discretization where Δx_1 and Δx_2 are the uniform nodal spacing in the 1- and 2-directions, respectively, β_i is a random number with range $-0.1 \leq \beta_i \leq 0.1$ for node i, and $\mathbf{x}_i'$ is the perturbed nodal position vector. The nodal point location perturbation is performed for every refined discretization in the convergence study. The results shown in Table VIII demonstrate a similar convergence behavior to that of uniform discretization.

TABLE VII. The total errors for $L = 10^2$, $c = 1.6$, and $N_c = 325$ by using the Newton iteration scheme.

	nth iteration					
	$n = 7$	$n = 8$	$n = 9$	$n = 10$	$n = 11$	$n = 12$
$\|u - \bar{u}_n\|_{\infty,\Omega}$	1.23	3.03 (−1)	2.50 (−2)	1.69 (−4)	1.13 (−5)	1.31 (−5)
$\|u - \bar{u}_n\|_{0,\Omega}$	1.03	2.25 (−1)	1.73 (−2)	1.16 (−4)	6.87 (−6)	6.87 (−6)
$\|u - \bar{u}_n\|_{1,\Omega}$	2.55	5.49 (−1)	4.21 (−2)	2.83 (−4)	3.10 (−5)	3.10 (−5)

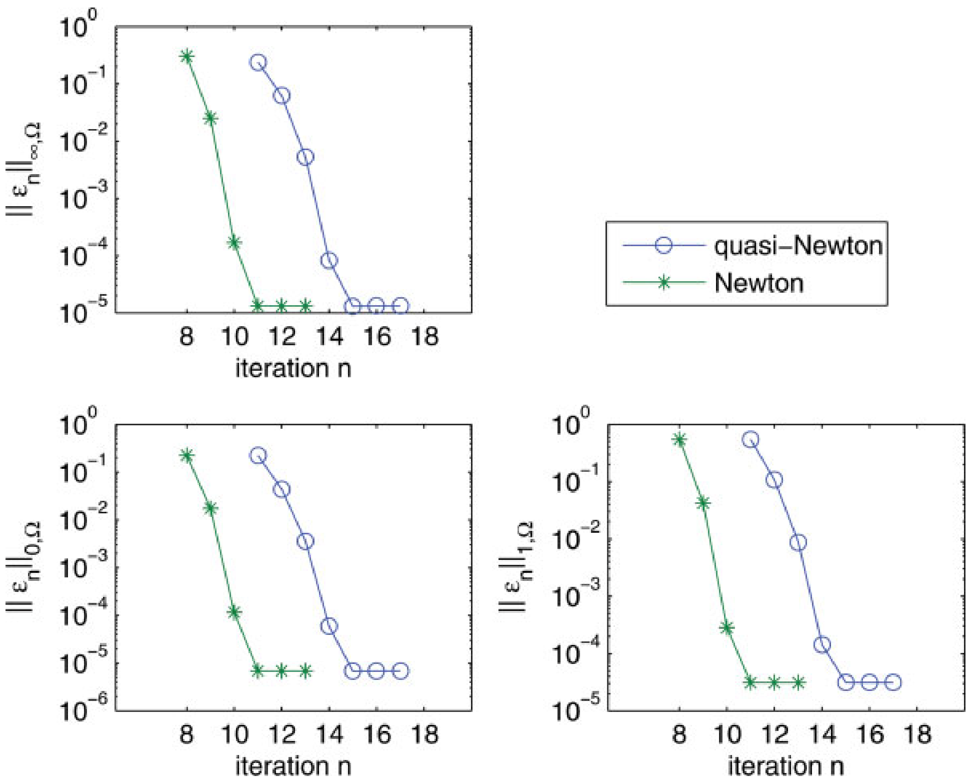

FIG. 4. The total errors in different norms versus iteration number for quasi-Newton and Newton iteration schemes, where ε_n denotes the error $u - \bar{u}_n$. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

V. CONCLUSION REMARKS

The present work is aimed at providing a convergence analysis and implementation schemes of the radial basis collocation method in conjunction with the quasi-Newton iteration for solving semilinear elliptic problems. The main results of this work indicate that there exists an exponential convergence with respect to the number and the shape factor of the radial basis functions, and a superlinear convergence with respect to the number of iteration. The numerical results by

TABLE VIII. The error norms and condition numbers for $c = 1.6$ and $N_c = 13 \times 25$ by using quasi-Newton iteration scheme with random set of source and collocation points.

	$L = N^2$			
	6^2	8^2	10^2	12^2
$\|u - \bar{u}_n\|_{\infty,\Omega}$	4.56 (−3)	1.41 (−4)	1.30 (−5)	1.81 (−6)
$\|u - \bar{u}_n\|_{0,\Omega}$	1.19 (−3)	5.56 (−5)	6.74 (−6)	9.51 (−7)
$\|u - \bar{u}_n\|_{1,\Omega}$	9.09 (−3)	4.58 (−4)	2.90 (−5)	2.98 (−6)
Cond.	4.23 (5)	1.18 (8)	4.13 (9)	1.40 (9)
No. of iterations	15	15	16	17

using this iteration scheme (4.3) agree well with the analytical results provided in Section 3. We summarize the merits for such numerical results as follows.

1. The source points and collocation points may be scattered in a rather arbitrary fashion. However, the number of collocation points should be chosen to be much larger than the number of source points. Such a scheme is not confined by problems with regular domain. This is a meshfree technique with considerable flexibilities, and it can be easily applied to higher dimensions.
2. The proposed method provides high accuracy. It can be seen from numerical results that the errors exhibit exponential convergence rates. Typical mesh-based approaches, for example, finite difference or finite element methods, offer algebraic convergence rates.
3. The simplicity of the proposed method allows easy computer coding. It is known that the mesh-based methods often involve time-consuming tasks in the construction of *a good quality mesh*, since the quality of mesh geometry plays an important role in the accuracy of numerical solution. The proposed method offers significant simplicity in its discretization.
4. Newton iteration scheme is commonly used in nonlinear problems due to its fast convergence rate. Although the convergence rate of 1.62 of the quasi-Newton iteration scheme is slightly slower than rate of 2 in the Newton iteration scheme, the former is more stable and less sensitive to the initial guess.
5. The analysis made in Subsection B of Section 3 is a general framework for analysis regardless of iteration method used. Lemma 3.1 can be extended to finite element or finite difference combined with any iteration methods discussed in this work.

The numerical experiments indicate that an adaptive $\delta - c$ scheme can be considered to maximize the effectiveness of the proposed method in solving partial differential equations. To achieve an effective adaptive computation, an a-posteriori error estimator is needed. The extension of the proposed radial basis collocation method to other nonlinear problems, such as the parameterized nonlinear problems [4], will appear elsewhere.

The support for this work by the Mathematics Division of the National Center for Theoretical Sciences (Taiwan Republic of China) to the first author and the support by Lawrence Livemore National Laboratory (U.S.A.) to the second author are gratefully acknowledged. The first author would also like to thank Professor Jinn-Liang Liu of National University of Kauhsiung for his inspiration of this work.

References

1. C. V. Pao, Block monotone iterative methods for numerical solutions of nonlinear elliptic equations, Numer Math 72 (1995), 239–262.
2. G. Saldanha, Technical note: a furth-order finite difference scheme for a system of 2D nonlinear elliptic partial differential equation, Numer Methods Partial Differential Equations 17 (2001), 43–53.
3. X. Liu, Y. S. Wang, and X. Ji, Montone iterations for numerical solutions of nonlinear elliptic partial differential equations, Appl Math Comput 50 (1992), 59–91.
4. F. Brezz, J. Rappaz, and P. A. Raviart, Finite dimensional approximation of nonlinear problems, I. Branches of nonsingular solutions, Numer Math 36 (1980), 1–25.
5. F. Brezz, J. Rappaz, and P. A. Raviart, Finite dimensional approximation of nonlinear problems, II. Limit points, Numer Math 37 (1981), 1–28.
6. F. Brezz, J. Rappaz, and P. A. Raviart, Finite dimensional approximation of nonlinear problems, III. Simple bifurcation points, Numer Math 38 (1981), 1–30.

7. Y. Deng, G. Chen, W.-M. Ni, and J. Zhou, Boundary element monotone iteration scheme for semilinear elliptic partial differential equations, Math Comput 65 (1996), 943–982.

8. R. A. Gingold and J. J. Monaghan, Smoothed particle hydrodynamics: theory and application to non-spherical stars, Mon Not R Astron Soc 181 (1977), 375–389.

9. W. K. Liu, S. Jun, S. Li, J. Adee, and T. Belytschko, Reproducing kernel particle methods for structural dynamics, Int J Numer Methods Eng 38 (1995), 1655–1679.

10. J. S. Chen, C. Pan, C. T. Wu, and W. K. Liu, Reproducing kernel particle methods for large deformation analysis of nonlinear structures, Comput Methods Appl Mech Eng 139 (1996), 195–227.

11. B. Nayroles, G. Touzot, and P. Villon, Generalizing the finite element method: diffuse approximation and diffuse elements, Comput Mech 10 (1992), 307–318.

12. T. Belytschko, Y. Y. Lu, and L. Gu, Element-free Galerkin methods, Int J Numer Methods Eng 37 (1994), 229–256.

13. Y. Y. Lu, T. Belytschko, and L. Gu, A new implementation of the element free Galerkin method, Comput Methods Appl Mech Eng 113 (1994), 397–414.

14. S. N. Atluri and T. L. Zhu, The meshless local Petrov–Galerkin (MLPG) approach for solving problems in elasto-statics, Comput Mech 25 (2000), 169–179.

15. E. Onate, S. Idelsohn, O. C. Zienkiewicz, and R. L. Taylor, A finite point method in computational mechanics: applications to convective transport and fluid flow, Int J Numer Methods Eng 39 (1996), 3839–3866.

16. J. M. Melenk and I. Babuska, The partition of unity finite element method: basic theory and applications, Comput Methods Appl Mech Eng 139 (1996), 289–314.

17. I. Babuska and J. M. Melenk, The partition of unity method, Int J Numer Methods Eng 40 (1997), 727–758.

18. C. A. Duarte and J. T. Oden, An h-p adaptive method using clouds, Comput Methods Appl Mech Eng 139 (1996), 237–262.

19. J. T. Oden, C. A. Duarte, and O. C. Zienkiewicz, A new cloud-based hp finite element method, Comput Methods Appl Mech Eng 153 (1998), 117–126.

20. S. De and K. J. Bathe, The method of finite spheres, Comput Mech 25 (2000), 329–345.

21. N. Sukumar, B. Moran, and T. Belytschko, The natural element method in solid mechanics, Int J Numer Methods Eng 43 (1998), 839–887.

22. E. J. Kansa, Multiquadrics—a scattered data approximation scheme with applications to computational fluid-dynamics. I. Surface approximations and partial derivatives, Comput Math Appl 19 (1992), 127–145.

23. E. J. Kansa, Multiquadrics—a scattered data approximation scheme with applications to computational fluid-dynamics. II. Solutions to parabolic, hyperbolic and elliptic partial differential equations, Comput Math Appl 19 (1992), 147–161.

24. T. J. R. Hughes, J. A. Cottrell, and Y. Bazilevs, Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement, Comput Methods Appl Mech Eng 194 (2005), 4135–4195.

25. S. H. Park and S. K. Youn, The least-squares meshfree method, Int J Numer Methods Eng 52 (2001), 997–1012.

26. X. Zhang, X.-H. Liu, K.-Z. Song, and M.-W. Lu, Least-squares collocation meshless method, Int J Numer Methods Eng 51 (2001), 1089–1100.

27. J. S. Chen, C. T. Wu, S. Yoon, and Y. You, A stabilized conforming nodal integration for Galerkin meshfree methods, Int J Numer Methods Eng 50 (2001), 435–466.

28. H. Y. Hu and Z. C. Li, Collocation methods for Poisson's equation, Comput Methods Appl Mech Eng 195 (2006), 4139–4160.

29. H. Y. Hu, J. S. Chen, and W. Hu, Weighted radial basis collocation method for boundary value problems, Int J Numer Methods Eng 69 (2007), 2736–2757.

30. L. Ling and E. J. Kansa, Preconditioning for radial basis functions with domain decomposition, Math Comput Model 40 (2004), 1413–1427.

31. H. Y. Hu, Z. C. Li, and A. H.-D. Cheng, Radial basis collocation method for elliptic equations, Comput Math Appl 50 (2005), 289–320.

32. W. R. Madych, Miscellaneous error bounds for multiquadric and related interpolatory, Comput Math Appl 24 (1992), 121–138.

33. J. Yoon, Interpolation by radial basis functions on Sobolev space, J Approx Theory 112 (2001), 1–15.

34. R. Franke and R. Schaback, Solving partial differential equations by collocation using radial functions, Appl Math Comput 93 (1998), 73–82.

35. Y. C. Hon and X. Z. Mao, An efficient numerical scheme for Burgers' equation, Appl Math Comput 95 (1998), 37–50.

36. H. Wendland, Meshless Galerkin methods using radial basis functions, Math Comput 68 (1999), 1521–1531.

37. Y. C. Hon, A quasi-radial basis functions method for American options pricing, Comput Math Appl 43 (2002), 513–524.

38. A. I. Shestakov, J. L. Milovich, and A. Noy, Using pseudo transient continuation and the finite element method to solve the nonlinear Poisson–Boltzmann equation, In Compact modeling and model order reduction (chapter 3), In Nanotech 2001 Vol. 1 (Technical Proccedings of the 2001 International Conference of Modeling and Simulation of Microsystems), Nano Science and Technology Institute, 2001, pp. 39–43.

39. A. Sayyed-Ahamad, K. Tuncay, and P. J. Ortoleva, Efficient solution technique for solving the Poisson–Boltzmann, J Comput Chem 25 (2004), 1068–1074.

40. N. Baker, M. Holst, and F. Wang, Adaptive multilevel finite element solution of the Poisson–Boltzmann equation. II. Refinement at solvent-accessible surface in biomolecular systems, J Comput Chem 21 (2000), 1343–1352.

41. M. Holst, N. Baker, and F. Wang, Adaptive multilevel finite element solution of the Poisson–Boltzmann equation. I. Algorithms examples, J Comput Chem 21 (2000), 1319–1342.

42. G. H. Golub and C. F. Van Loan, Matrix computations, The Johns Hopkins University Press, Baltimore, MD, 1996.