Error Analysis of Collocation Method Based on Reproducing Kernel Approximation

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Received 4 May 2009; accepted 31 August 2009
Published online 6 November 2009 in Wiley Online Library (wileyonlinelibrary.com).
DOI 10.1002/num.20539

Solving partial differential equations (PDE) with strong form collocation and nonlocal approximation functions such as orthogonal polynomials, trigonometric functions, and radial basis functions exhibits exponential convergence rates; however, it yields a full matrix and suffers from ill conditioning. In this work, we discuss a reproducing kernel collocation method, where the reproducing kernel (RK) shape functions with compact support are used as approximation functions. This approach offers algebraic convergence rate, but the method is stable like the finite element method. We provide mathematical results consisting of the optimal error estimation, upper bound of condition number, and the desirable relationship between the number of nodal points and the number of collocation points. We show that using RK shape function for collocation of strong form, the degree of polynomial basis functions has to be larger than one for convergence, which is different from the condition for weak formulation. Numerical results are also presented to validate the theoretical analysis. Numer Methods Partial Differential Eq 27: 554–580, 2011

Keywords: collocation method; convergence; inverse inequality; reproducing kernel approximation; stability

I. INTRODUCTION

The development of meshfree methods can be traced back from two branches, one based on Galerkin weak formulation and the other based on strong formulation. Compactly supported moving least-squares (MLS) approximation [1] and reproduction kernel (RK) approximation [2] are most commonly used in the methods based on weak formulation. On the other hand, nonlocal radial basis functions [3, 4] is widely used in the strong form approach. Methods developed under weak form [5–16] are typically well-conditioned owing to the use of compactly

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Contract grant sponsors: National Science Council (Taiwan, Republic of China), Lawrence Livermore National Laboratory (USA)

supported approximation functions, and they exhibit algebraic convergence due to employment of monomial basis functions. It is noteworthy that the seminal work of partition of unity by Babuska et al. [17] provided the foundation of generalized finite element and meshfree method developments. However, the need of quadrature rules in the domain integration of weak form and the complications in imposing Dirichlet boundary conditions have caused considerable complexity and high computational cost to this class of methods. On the other hand, methods based on strong form collocation, such as the radial basis collocation method [18–26], eliminate the need of quadrature rules and simplify the imposition of Dirichlet boundary conditions in addition to the attractive exponential convergence property. However, these methods are overshadowed by their fully dense and ill-conditioned discrete equation due to the use of nonlocal radial basis functions.

Efforts have been devoted to take complementary advantages between the two classes of methods described earlier. Among them, the most noticeable work is the finite point method by Onate et al. [27], where they introduced compactly supported MLS approximation in the strong form with direct collocation. Since this approach requires taking higher order derivatives of the MLS approximation functions, several enhanced methods have been proposed based on RK approximation with derivative reproducing conditions [28] or diffuse derivatives [29]. The work in [25, 30] introduced a concept commonly used in the radial basis function method, where the number of collocation points used to enforce zero residual of the strong form and boundary conditions is more than the number of nodal points for improved accuracy of the strong form collocation method. This yields an over-determined system and is then solved by the least-squares method. On the other hand, methods based on weak formulation have been modified by introducing nodal integration with stabilization to avoid quadrature rules. However, nodal integration of weak form yields rank instability, and this rank instability has been addressed by adding a least-squares residual of the equilibrium equation in the nodally integrated weak form as a stabilization [31], by introducing a correction of nodal integration for patch test in conjunction with a least-squares stabilization [32], by embedding stress points under the smoothed particle hydrodynamics framework [33], or by introducing a stabilized conforming nodal integration [34, 35] based on a strain smoothing to avoid taking shape function derivatives at nodal points for stability and to satisfy integration constraints for passing patch test.

Although the strong form collocation in conjunction with RK shape functions for numerical solution of partial differential equations (PDE) has been introduced before as discussed earlier, this is the first work to investigate the convergence and stability of this approach. The objective of this study is to estimate error bounds and condition numbers of the methods developed based on strong form collocation with reproducing kernel approximation, as many methods described earlier resemble the essential features of this method. We term this class of method the reproducing kernel collocation method (RKCM). A crucial point in the analysis discussed herein is to realize that the over-determined discrete equation obtained by collocation of strong form and solved by least-squares method is equivalent to minimization of least-squares functional with quadrature rules [24, 25]. This property allows us to use inverse inequality of reproducing kernel approximation in conjunction with partition of unity properties for the error analysis of RKCM. We also obtain optimal relationship for the number of nodal points and the number of collocation points. An important result extracted from this analysis is that the degree of monomial bases in the reproducing kernel approximation has to be greater than one for convergence in RKCM, and this is different from the convergence condition for weak form based method such as reproducing kernel particle method (RKPM). The analysis reveals that the condition number of RKCM increases algebraically, similar to the conditioning of the finite element method.

The remaining part of this article is organized as follows. An introduction to reproducing kernel function and its derivatives, approximation error, and inverse inequality are given in Section II. In

Section III, the general collocation framework as well as implementation issues and complexity analysis are presented. The analysis of convergence properties and stability of RKCM are given in Section IV. Section V presents the numerical experiments to validate the results of the error analysis. Conclusions are summarized in Section VI.

II. REPRODUCING KERNEL (RK) APPROXIMATION

A. One-Dimensional RK Shape Functions

Let function $u(x)$ be approximated by

$$u^r(x) = \sum_{I=1}^{N_P} \Psi_I(x) u_I, \quad \forall x \in \Omega \subset \mathcal{R}^1, \tag{2.1}$$

where $\Psi_I(x)$ are shape functions centered at x_I, and u_I are the coefficients to be sought. The construction of the basis functions is based on a set of points in the domain,

$$\mathcal{X}_p = \{x_1, x_2, \ldots, x_{N_P}\} \subseteq \bar{\Omega}, \tag{2.2}$$

where $\bar{\Omega} = \Omega \cup \partial\Omega$, N_P is the number of the discrete points, and we assume that the maximal nodal distance is h.

In RK approximation, the shape functions $\Psi_I(x)$ are constructed by the multiplication of two functions

$$\Psi_I(x) = C(x; x - x_I)\phi_a(x - x_I), \tag{2.3}$$

where $\phi_a(x - x_I)$ is a kernel function that defines the smoothness of approximation with a compact support, ω_I, measured by a and $C(x; x - x_I)$ is called the correction function used to reproduce of polynomial or other functions. For flexibility, we allow the support size a to be dependent on I. Examples, the cubic B-spline,

$$\phi_a(z) = \begin{cases} \frac{2}{3} - 4z^2 + 4z^3, & \text{for } 0 \le z < \frac{1}{2}, \\ \frac{4}{3} - 4z + 4z^2 - \frac{4}{3}z^3, & \text{for } \frac{1}{2} \le z < 1, \\ 0, & \text{for } z \ge 1, \end{cases} \tag{2.4}$$

and the quintic B-spline,

$$\phi_a(z) = \begin{cases} \frac{11}{20} - \frac{9z^2}{2} + \frac{81z^4}{4} - \frac{81z^5}{4}, & \text{for } 0 \le z < \frac{1}{3}, \\ \frac{17}{40} + \frac{15z}{8} - \frac{63z^2}{4} + \frac{135z^3}{4} - \frac{243z^4}{8} - \frac{81z^5}{8}, & \text{for } \frac{1}{3} \le z < \frac{2}{3}, \\ \frac{81}{40} - \frac{81z}{8} + \frac{81z^2}{4} - \frac{81z^3}{4} - \frac{81z^4}{8} - \frac{81z^5}{40}, & \text{for } \frac{2}{3} \le z < 1, \\ 0, & \text{for } z \ge 1, \end{cases} \tag{2.5}$$

where $z = \frac{|x-x_I|}{a}$. For reproduction of complete p-th order polynomial, the correction function $C(x; x-x_I)$ is formed by a set of polynomial basis as

$$C(x; x-x_I) = \sum_{i=0}^{p} b_i(x)(x-x_I)^i, \quad p \geq 0. \tag{2.6}$$

The coefficient are to be obtained by satisfying the p-th order reproducing conditions [16, 34]

$$\sum_{I=1}^{N_P} \Psi_I(x) x_I^i = x^i, \quad i = 0, 1, \ldots, p, \tag{2.7}$$

i.e.,

$$\sum_{I=1}^{N_P} C(x; x-x_I)\phi_a(x-x_I) x_I^i = x^i, \quad i = 0, 1, \ldots, p. \tag{2.8}$$

Equation (2.8) is equivalent to

$$\sum_{I=1}^{N_P} C(x; x-x_I)\phi_a(x-x_I)(x-x_I)^i = \delta_{i0}, \quad i = 0, 1, \ldots, p, \tag{2.9}$$

where δ_{ij} is the Kronecker delta. Equation (2.9) can be rewritten in a vector form

$$\sum_{I=1}^{N_P} C(x; x-x_I)\phi_a(x-x_I)\mathbf{h}(x-x_I) = \mathbf{h}(0), \tag{2.10}$$

where

$$\mathbf{h}(z) = [1, z, z^2, \ldots, z^p]^{\mathrm{T}}, \tag{2.11}$$

and

$$\mathbf{h}(0) = [1, 0, \ldots, 0]^{\mathrm{T}}. \tag{2.12}$$

Denote the correction function in (2.6) in a vector form

$$C(x; x-x_I) = \mathbf{h}^{\mathrm{T}}(x-x_I)\mathbf{b}(x), \tag{2.13}$$

where

$$\mathbf{b}(x) = [b_0(x), b_1(x), \ldots, b_p(x)]^{\mathrm{T}}. \tag{2.14}$$

Substituting (2.13) into (2.10), we obtain

$$\sum_{I=1}^{N_P} \mathbf{h}(x-x_I)\phi_a(x-x_I)\mathbf{h}^{\mathrm{T}}(x-x_I)\mathbf{b}(x) = \mathbf{h}(0). \tag{2.15}$$

We rewrite it as

$$\mathbf{M}(x)\mathbf{b}(x) = \mathbf{h}(0), \tag{2.16}$$

where

$$\mathbf{M}(x) = \sum_{I=1}^{N_P} \mathbf{h}(x - x_I)\mathbf{h}^{\mathrm{T}}(x - x_I)\phi_a(x - x_I). \tag{2.17}$$

Here, $\mathbf{M}(x)$ is called the moment matrix. It follows from (2.16) that

$$\mathbf{b}(x) = \mathbf{M}^{-1}(x)\mathbf{h}(0). \tag{2.18}$$

Therefore, the shape function $\Psi_I(x)$ is obtained as

$$\Psi_I(x) = \mathbf{h}^{\mathrm{T}}(0)\mathbf{M}^{-1}(x)\mathbf{h}(x - x_I)\phi_a(x - x_I). \tag{2.19}$$

The function $\Psi_I(x)$ is called reproducing kernel (RK) shape function.

Remark 2.1. At any position $x \in \Omega$, a necessary condition for moment matrix $\mathbf{M}(x)$ to be invertible is that x needs to be covered by at least $p+1$ RK shape functions. This is the restriction on how the support size of the kernel functions should be chosen. In this article, we choose the support size to be proportional to nodal distance h, say $a = (p + \delta)h$, $\delta > 0$ for any point distribution. For computational efficiency and keeping well conditioning of discrete equations, we also assume that any position x is covered by at most κ shape functions, $p + 1 \leq \kappa \ll N_p$.

Remark 2.2. The complexity of the RK shape function includes the formation of matrix $\mathbf{M}(x)$, the calculation of $\mathbf{M}^{-1}(x)$, the construction of correction function $C(x; x - x_I)$, and the construction of shape function $\Psi_I(x)$. The total operation counts for the formation of $\Psi_I(x)$ are given as follows [36]

$$\begin{aligned} \text{M/D}: &\quad s^3 + (2\kappa + 1)s^2 + s + 1, \\ \text{A/S}: &\quad s^3 + (\kappa - 2)s^2 + s - 1. \end{aligned} \tag{2.20}$$

The abbreviations M/D and A/S are multiplication/division and addition/subtraction, respectively, s is the dimension of the vector $\mathbf{h}(x - x_I)$, and κ is the value satisfying overlapping condition in Remark 2.1.

B. Multidimensional RK Approximation

The basis function discussed in the previous section can be extended to multidimension. Figure 1 demonstrates a two-dimensional RK discretization, where the geometry of the compact supports can be rectangular or circular shape.

In the three-dimensional approximation, let function $u(\mathbf{x})$ be approximated by

$$u^r(\mathbf{x}) = \sum_{I=1}^{N_P} \Psi_I(\mathbf{x})u_I, \quad \forall \mathbf{x} \in \Omega \subset \mathcal{R}^3, \tag{2.21}$$

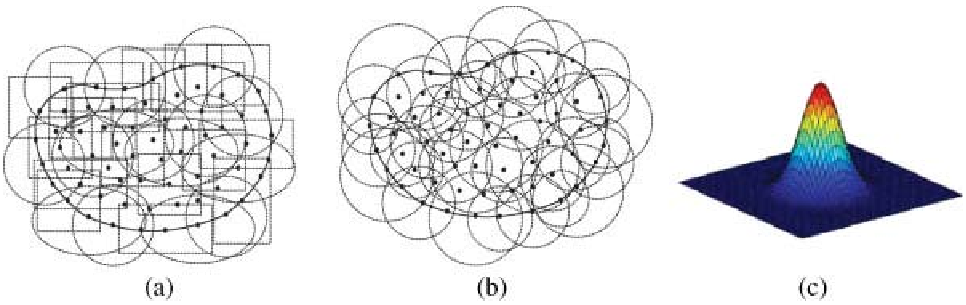

FIG. 1. (a,b) RK discretization and compact support in two-dimension and (c) RK shape function. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

where $\mathbf{x} = (x_1, x_2, x_3)^{\mathrm{T}}$ and shape functions are expressed in the form of

$$\Psi_I(\mathbf{x}) = C(\mathbf{x}; \mathbf{x} - \mathbf{x}_I)\phi_a(\mathbf{x} - \mathbf{x}_I), \tag{2.22}$$

where the center $\mathbf{x}_I = (x_{1I}, x_{2I}, x_{3I})^{\mathrm{T}}$, and the correction function is

$$\begin{aligned} C(\mathbf{x}; \mathbf{x} - \mathbf{x}_I) &= \sum_{i+j+k=0}^{p} (x_1 - x_{1I})^i (x_2 - x_{2I})^j (x_3 - x_{3I})^k b_{ijk}(\mathbf{x}) \\ &=: \mathbf{h}^{\mathrm{T}}(\mathbf{x} - \mathbf{x}_I)\mathbf{b}(\mathbf{x}), \end{aligned} \tag{2.23}$$

and

$$\mathbf{h}(\mathbf{x} - \mathbf{x}_I) = [1, x_1 - x_{1I}, x_2 - x_{2I}, x_3 - x_{3I}, \ldots, (x_3 - x_{3I})^p]^{\mathrm{T}}. \tag{2.24}$$

The shape functions in (2.22) satisfy the following p-th order reproducing conditions

$$\sum_{I=1}^{N_P} \Psi_I(\mathbf{x}) x_{1I}^i x_{2I}^j x_{3I}^k = x_1^i x_2^j x_3^k, \quad i + j + k = 0, 1, \ldots, p. \tag{2.25}$$

This approximation satisfies the partition of unity when $i + j + k = 0$, i.e.,

$$\sum_{I=1}^{N_P} \Psi_I(\mathbf{x}) = 1. \tag{2.26}$$

The coefficient vector $\mathbf{b}(\mathbf{x})$ in (2.23) are obtained by satisfying Eq. (2.25). Thus, we obtain a three-dimensional RK shape function as follows

$$\Psi_I(\mathbf{x}) = \mathbf{h}^{\mathrm{T}}(\mathbf{0})\mathbf{M}^{-1}(\mathbf{x})\mathbf{h}(\mathbf{x} - \mathbf{x}_I)\phi_a(\mathbf{x} - \mathbf{x}_I), \tag{2.27}$$

where

$$\mathbf{M}(\mathbf{x}) = \sum_{I=1}^{N_P} \mathbf{h}(\mathbf{x} - \mathbf{x}_I)\mathbf{h}^{\mathrm{T}}(\mathbf{x} - \mathbf{x}_I)\phi_a(\mathbf{x} - \mathbf{x}_I). \tag{2.28}$$

The complexity for constructing the multidimensional shape function in (2.27) is the same as that stated in Remark 2.2; they are different only in space dimension. The properties of the reproducing kernel shape functions are summarized as follows.

(1) The construction of shape function $\Psi_I(\mathbf{x})$ is based on a set of points, and it does not rely on a mesh.
(2) The shape function $\Psi_I(\mathbf{x})$ has the same compact support as that of the kernel function $\phi_a(\mathbf{x})$.
(3) For the moment matrix $\mathbf{M}(\mathbf{x})$ to be nonsingular, any position $\mathbf{x} \in \Omega$ needs to be covered by at least s non collinear or non coplanar kernel functions, where $s = (p+3)!/p!3!$ in three dimension.
(4) The shape function $\Psi_I(\mathbf{x})$ does not possess the Kronecker delta property, thus the coefficient is not the nodal value, i.e.,

$$\Psi_I(\mathbf{x}_J) \neq \delta_{IJ}, \quad \text{and} \quad u^r(\mathbf{x}_I) \neq u_I.$$

(5) The smoothness of the shape function $\Psi_I(\mathbf{x})$ is the same as the smoothness of the kernel function $\phi_a(\mathbf{x})$ if monomial basis functions are used.

C. Derivatives of RK Shape Function

Following the construction of the shape functions $\Psi_I(x)$ in (2.22), the first-order partial derivative is given as

$$\Psi_{I,x_i}(\mathbf{x}) = C_{,x_i}(\mathbf{x}-\mathbf{x}_I)\phi_a(\mathbf{x}-\mathbf{x}_I) + C(\mathbf{x}-\mathbf{x}_I)\phi_{a,x_i}(\mathbf{x}-\mathbf{x}_I), \tag{2.29}$$

where $i = 1, 2, \ldots, d$, and d is the space dimension. The second-order derivative is of the form

$$\Psi_{I,x_i^2}(\mathbf{x}) = C_{,x_i^2}(\mathbf{x}-\mathbf{x}_I)\phi_a(\mathbf{x}-\mathbf{x}_I) + 2C_{,x_i}(\mathbf{x}-\mathbf{x}_I)\phi_{a,x_i}(\mathbf{x}-\mathbf{x}_I) + C(\mathbf{x}-\mathbf{x}_I)\phi_{a,x_i^2}(\mathbf{x}-\mathbf{x}_I), \tag{2.30}$$

wherein the derivative of correction function is given as

$$C_{,x_i} = \mathbf{h}^{\mathrm{T}}(\mathbf{0})\mathbf{M}^{-1}_{,x_i}(\mathbf{x})\mathbf{h}(\mathbf{x}-\mathbf{x}_I) + \mathbf{h}^{\mathrm{T}}(\mathbf{0})\mathbf{M}^{-1}(\mathbf{x})\mathbf{h}_{,x_i}(\mathbf{x}-\mathbf{x}_I), \tag{2.31}$$

and

$$C_{,x_i^2} = \mathbf{h}^{\mathrm{T}}(\mathbf{0})\mathbf{M}^{-1}_{,x_i^2}(\mathbf{x})\mathbf{h}(\mathbf{x}-\mathbf{x}_I) + 2\mathbf{h}^{\mathrm{T}}(\mathbf{0})\mathbf{M}^{-1}_{,x_i}(\mathbf{x})\mathbf{h}_{,x_i}(\mathbf{x}-\mathbf{x}_I) + \mathbf{h}^{\mathrm{T}}(\mathbf{0})\mathbf{M}^{-1}(\mathbf{x})\mathbf{h}_{,x_i^2}(\mathbf{x}-\mathbf{x}_I). \tag{2.32}$$

The terms ϕ_{a,x_i} and ϕ_{a,x_i^2} are obtained directly by taking differentiation on the kernel function ϕ_a, and the derivatives $\mathbf{M}_{,x_i}$ and $\mathbf{M}_{,x_i^2}$ are obtained by taking differentiation on moment matrix $\mathbf{M}$; furthermore, the inversions of the derivatives are obtained by using the following relationships

$$\mathbf{M}^{-1}_{,x_i} = -\mathbf{M}^{-1}\mathbf{M}_{,x_i}\mathbf{M}^{-1}, \tag{2.33}$$

and

$$\mathbf{M}^{-1}_{,x_i^2} = -\mathbf{M}^{-1}\{\mathbf{M}_{,x_i^2}\mathbf{M}^{-1} + 2\mathbf{M}_{,x_i}\mathbf{M}^{-1}_{,x_i}\}. \tag{2.34}$$

The complexity of RK shape function derivatives are listed in the following remark.

Remark 2.3. The operation counts for the first-order derivative of RK shape function are given as follows

$$\begin{aligned} \mathrm{M/D}: &\quad 3s^3 + (8\kappa + 4)s^2 + 3s + 2, \\ \mathrm{A/S}: &\quad 3s^3 + (4\kappa - 5)s^2 + s + 1. \end{aligned} \tag{2.35}$$

For the second-order derivative of RK shape function, the operation counts are

$$\begin{aligned} \mathrm{M/D}: &\quad 6s^3 + (20\kappa + 12)s^2 + 6s + 4, \\ \mathrm{A/S}: &\quad 6s^3 + (10\kappa - 11)s^2 + s + 12, \end{aligned} \tag{2.36}$$

where $s = (p + d)!/(p!d!)$, and d is space dimension.

D. The Interpolation Estimate and Inverse Inequality

Based on the choice of support size stated in Remark 2.1, we assume a quasi-uniform distribution of the discrete points for approximation of a function or for solving PDE as defined below.

Defnition 2.1. *A set of points is said to be quasi-uniform distribution if there exists two constants c_0 and c_1 such that*

$$c_0 \leq \frac{a_I}{a_J} \leq c_1, \quad \forall I, J, \tag{2.37}$$

where a_I and a_J denotes the support size of kernel functions centered at $\mathbf{x}_I$ and $\mathbf{x}_J$, respectively, and the support sizes are proportional to nodal distance h.

Denote V_k, the finite dimensional collection of the RK shape function defined in (2.27),

$$V_k = \mathrm{span}\{\Psi_1(\mathbf{x}), \Psi_2(\mathbf{x}), \ldots, \Psi_{N_p}(\mathbf{x})\}, \tag{2.38}$$

and $\|\cdot\|_{\ell,\omega_I}$, $\|\cdot\|_{\ell,\Omega}$,$|\cdot|_{\ell,\omega_I}$, and $|\cdot|_{\ell,\Omega}$ the Sobolev norms and seminorms.

Assume the kernel function $\phi(\mathbf{x} - \mathbf{x}_I)$ is ℓ-times differentiable. The bounds for the RK shape function and its derivatives are given in the lemma below [16, 34]. They are used for the error estimation.

Lemma 2.1. *Consider the quasi-uniform points distribution, there exist two constants C_0 and C_1 such that*

$$\max_{1\leq I\leq N_P} \|\Psi_I\|_\infty \leq C_0, \tag{2.39}$$

$$\max_{1\leq I\leq N_P} \max_{|\alpha|=\ell} \|D^\alpha \Psi_I\|_\infty \leq C_1 a^{-\ell}, \tag{2.40}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ is a multi-index, α-th derivative operator is defined as

$$D^\alpha f(\mathbf{x}) = \frac{\partial^{|\alpha|} f(\mathbf{x})}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_d^{\alpha_d}}, \tag{2.41}$$

and $|\alpha| = \sum_{i=1}^d \alpha_i$ is the length of α.

We have the following results concerning the global interpolation error.

Lemma 2.2. *Consider a quasi-uniform points distribution. For $u^r \in V_k$, there exists a local estimation*

$$\|u - u^r\|_{\ell,\omega_I} \leq C a^{p+1-\ell} |u|_{p+1,\omega_I}, \quad \forall \ell \geq 0, \tag{2.42}$$

where a is the maximal support size in $\bar{\Omega}$, p is the reproducing degree of u^r, and C is a generic constant.

Moreover, we obtain a global estimation

$$\|u - u^r\|_{\ell,\Omega} \leq C\kappa a^{p+1-\ell} |u|_{p+1,\Omega}, \quad \forall \ell \geq 0, \tag{2.43}$$

where κ is the value satisfying overlapping condition, and C is a generic constant.

Proof. The local error estimation in Sobolev norm can be found in [16,34]. The global error estimation is summarized as follows.

Using the property of partition of unity, we have

$$\begin{aligned}
\|u - u^r\|_{\ell,\Omega}^2 &= \left\| \sum_{I=1}^{N_P} \Psi_I (u - u_I) \right\|_{\ell,\Omega}^2 = \left\| D^\ell \sum_{I=1}^{N_P} \Psi_I (u - u_I) \right\|_{0,\Omega}^2 \\
&\leq 2 \left\| \sum_{I=1}^{N_P} (D^\ell \Psi_I)(u - u_I) \right\|_{0,\Omega}^2 + 2 \left\| \sum_{I=1}^{N_P} \Psi_I D^\ell (u - u_I) \right\|_{0,\Omega}^2 \\
&= 2 \int_\Omega \sum_{I=1}^{N_P} (D^\ell \Psi_I)^2 (u - u_I)^2 + 2 \int_\Omega \sum_{I=1}^{N_P} (\Psi_I)^2 (D^\ell (u - u_I))^2 \\
&\leq 2\kappa \int_{\omega_I} \sum_{I=1}^{N_P} (D^\ell \Psi_I)^2 (u - u_I)^2 + 2\kappa \int_{\omega_I} \sum_{I=1}^{N_P} (\Psi_I)^2 (D^\ell (u - u_I))^2 \\
&= 2\kappa \max_{1 \leq I \leq N_P} \|D^\ell \Psi_I\|_\infty^2 \sum_{I=1}^{N_P} \int_{\omega_I} (u - u_I)^2 \\
&\quad + 2\kappa \max_{1 \leq I \leq N_P} \|\Psi_I\|_\infty^2 \sum_{I=1}^{N_P} \int_{\omega_I} \left(D^\ell (u - u_I)\right)^2.
\end{aligned} \tag{2.44}$$

It follows from Lemmas 2.1 and 2.2 that

$$\begin{aligned}
\|u - u^r\|_{\ell,\Omega}^2 &\leq 2\kappa C_1^2 a^{-2\ell} \sum_{I=1}^{N_P} \|u - u^r\|_{0,\omega_I}^2 + 2\kappa C_0^2 \sum_{I=1}^{N_P} \|u - u^r\|_{\ell,\omega_I}^2 \\
&\leq 2\kappa C_1^2 a^{-2\ell} \kappa a^{2(p+1)} |u|_{p+1,\omega_I}^2 + 2\kappa C_0^2 \kappa a^{2(p+1)-2\ell} |u|_{p+1,\omega_I}^2 \\
&\leq C_2 \kappa^2 a^{2(p+1)-2\ell} |u|_{p+1,\omega_I}^2,
\end{aligned} \tag{2.45}$$

where $C_2 = \max\{2C_0^2, 2C_1^2\}$. Thus, we obtain

$$\|u - u^r\|_{\ell,\Omega} \leq C_3 \kappa a^{p+1-\ell} |u|_{p+1,\omega_I} \leq C\kappa a^{p+1-\ell} |u|_{p+1,\Omega}, \tag{2.46}$$

where C is a generic constant independent of the parameters a, κ, and p. ∎

In the following, we provide an inverse inequality that is critical to the further error and stability analysis.

Theorem 2.1. *Assume that the discrete points have quasi-uniform distribution, there exists a generic constant C such that*

$$\|v\|_{\ell,\Omega} \le C\kappa^{1/2} a^{-\ell} p^{2\ell} \|v\|_{0,\Omega}, \quad \forall \ell \ge 1, \tag{2.47}$$

where κ is the overlapping parameter, a is the kernel compact support parameter, and p is the reproducing order.

Proof. For simplicity, we consider the case in a two-dimensional setting here. Let v be an approximate function defined in a rectangular support ω_I,

$$\omega_I = \{(x_1, x_2) | x_{1I} - a_I \le x_1 \le x_{1I} + a_I, x_{2I} - b_I \le x_2 \le x_{2I} + b_I\}, \tag{2.48}$$

which the compact support is centered at (x_{1I}, x_{2I}) with dimension $2a_I \times 2b_I$. Denote a reference support by

$$\Box = \{(\xi, \eta) | -1 \le \xi \le 1, -1 \le \eta \le 1\}, \tag{2.49}$$

where the support is defined in reference coordinate system.

The support ω_I can be mapped onto $\Box$ by a linear transformation, $\mu = T_I(v)$, and the reference coordinates are defined as

$$\xi = \frac{1}{a_I}(x_1 - x_{1I}), \quad \eta = \frac{1}{b_I}(x_2 - x_{2I}), \tag{2.50}$$

so that

$$dx_1 = a_I d\xi, \quad dx_2 = b_I d\eta. \tag{2.51}$$

Moreover, we have

$$v(x_1, x_2)|_{\omega_I} = v(x_1(\xi), x_2(\eta)) = \mu(\xi, \eta)|_{\Box}. \tag{2.52}$$

and let

$$v_{x_1} = \frac{\partial v}{\partial x_1} = \frac{\partial \mu}{\partial \xi}\frac{d\xi}{dx_1} = \frac{\mu_\xi}{a_I}, \quad v_{x_2} = \frac{\partial v}{\partial x_2} = \frac{\partial \mu}{\partial \eta}\frac{d\eta}{dx_2} = \frac{\mu_\eta}{b_I}. \tag{2.53}$$

It follows that

$$\|v\|_{0,\omega_I}^2 = \int_{\omega_I} v^2 dx_1 dx_2 = a_I b_I \int_{\Box} \mu^2 d\xi d\eta = a_I b_I \|\mu\|_{0,\Box}^2. \tag{2.54}$$

Express a monomial with degree p in support $\Box$ as

$$\mu = \mu(\xi, \eta) = \sum_{i,j=0}^{p} b_{ij} \xi^i \eta^j. \tag{2.55}$$

The above can be expressed by the Legendre polynomials, and there exist the following inequalities [37]

$$\left\|\frac{\partial\mu}{\partial\xi}\right\|_{0,\square} = \|\mu_\xi\|_{0,\square} \le c_1 p^2 \|\mu\|_{0,\square}, \tag{2.56}$$

$$\left\|\frac{\partial\mu}{\partial\eta}\right\|_{0,\square} = \|\mu_\eta\|_{0,\square} \le c_2 p^2 \|\mu\|_{0,\square}, \tag{2.57}$$

and

$$|\mu|_{1,\square} \le c_3 p^2 \|\mu\|_{0,\square}, \quad \|\mu\|_{1,\square} \le c_4 p^2 \|\mu\|_{0,\square}. \tag{2.58}$$

For the function v defined in support ω_I, we obtain a relation as follows

$$\begin{aligned}
|v|^2_{1,\omega_I} &= \int_{\omega_I} \left(v^2_{x_1} + v^2_{x_2}\right) dx\,dy \\
&= a_I b_I \int_\square \left\{ \left(\frac{\mu_\xi}{a_I}\right)^2 + \left(\frac{\mu_\eta}{b_I}\right)^2 \right\} d\xi\, d\eta \\
&= \frac{b_I}{a_I}\|\mu_\xi\|^2_{0,\square} + \frac{a_I}{b_I}\|\mu_\eta\|^2_{0,\square} \le c_1 \frac{b_I}{a_I} p^4 \|\mu\|^2_{0,\square} + c_2 \frac{a_I}{b_I} p^4 \|\mu\|^2_{0,\square} \\
&\le \left(c_1 \frac{b_I}{a_I} + c_2 \frac{a_I}{b_I}\right) \frac{p^4}{a_I b_I} \|v\|^2_{0,\omega_I} \le \left(\frac{c_3}{a_I^2} + \frac{c_4}{b_I^2}\right) p^4 \|v\|^2_{0,\omega_I}.
\end{aligned} \tag{2.59}$$

We consider the case of $a_I = b_I = a$, the inequality in (2.59) leads to

$$|v|^2_{1,\omega_I} \le c_5 a^{-2} p^4 \|v\|^2_{0,\omega_I}, \quad \|v\|^2_{1,\omega_I} \le c_6 a^{-2} p^4 \|v\|^2_{0,\omega_I}. \tag{2.60}$$

Furthermore, the second partial derivatives of function v are given by

$$v_{x_1x_1} = \frac{\mu_{\xi\xi}}{a_I^2}, \quad v_{x_1x_2} = \frac{\mu_{\xi\eta}}{a_I b_I}, \quad v_{x_2x_2} = \frac{\mu_{\eta\eta}}{b_I^2}. \tag{2.61}$$

Similarly, there exist

$$|\mu_{\xi\xi}|_{0,\square} \le \bar{c}_1 p^4 \|\mu\|_{0,\square}, \quad |\mu_{\xi\eta}|_{0,\square} \le \bar{c}_2 p^4 \|\mu\|_{0,\square}, \quad |\mu_{\eta\eta}|_{0,\square} \le \bar{c}_3 p^4 \|\mu\|_{0,\square}. \tag{2.62}$$

Correspondingly, we have

$$\begin{aligned}
|v|^2_{2,\omega_I} &= \int_{\omega_I} \left(v^2_{x_1x_1} + 2v^2_{x_1x_2} + v^2_{x_2x_2}\right) dx\,dy \\
&= a_I b_I \int_\square \left\{ \left(\frac{\mu_{\xi\xi}}{a_I^2}\right)^2 + 2\left(\frac{\mu_{\xi\eta}}{a_I b_I}\right)^2 + \left(\frac{\mu_{\eta\eta}}{b_I^2}\right)^2 \right\} d\xi\, d\eta \\
&= \frac{b_I}{a_I^3}\|\mu_{\xi\xi}\|^2_{0,\square} + \frac{2}{a_I b_I}\|\mu_{\xi\eta}\|^2_{0,\square} + \frac{a_I}{b_I^3}\|\mu_{\eta\eta}\|^2_{0,\square} \\
&\le \left(\bar{c}_1 \frac{b_I}{a_I^3} + \bar{c}_2 \frac{2}{a_I b_I} + \bar{c}_3 \frac{a_I}{b_I^3}\right) p^8 \|\mu\|^2_{0,\square} \\
&\le \left(\bar{c}_4 \frac{b_I}{a_I^3} + \bar{c}_5 \frac{2}{a_I b_I} + \bar{c}_6 \frac{a_I}{b_I^3}\right) \frac{p^8}{a_I b_I} \|v\|^2_{0,\omega_I}
\end{aligned} \tag{2.63}$$

Considering the case of $a_I = b_I = a$, the inequality in (2.63) becomes

$$|v|^2_{2,\omega_I} \leq \bar{c}_7 a^{-4} p^8 \|v\|^2_{0,\omega_I}, \quad \|v\|^2_{2,\omega_I} \leq \bar{c}_8 a^{-4} p^8 \|v\|^2_{0,\omega_I}. \tag{2.64}$$

More generally, for any $\ell \geq 1$, we have the following result

$$\begin{aligned} \|v\|_{\ell,\Omega} &\leq C_1 \left\{ \sum_{I=1}^{N_P} \|v\|^2_{\ell,\omega_I} \right\}^{1/2} \leq C_2 \left\{ \kappa \|v\|^2_{\ell,\omega_I} \right\}^{1/2} \\ &\leq C_3 \sqrt{\kappa} a^{-\ell} p^{2\ell} \|v\|_{0,\omega_I} \leq C \sqrt{\kappa} a^{-\ell} p^{2\ell} \|v\|_{0,\Omega}, \end{aligned} \tag{2.65}$$

where $\ell = 1, 2, \ldots$. This proves (2.47). ■

III. REPRODUCING KERNEL COLLOCATION METHOD (RKCM)

A. Strong Form Collocation Method

Consider a Poisson problem

$$-\Delta u = f, \quad \text{in } \Omega, \tag{3.1}$$

$$u_\nu = q_1, \quad \text{on } \Gamma_N, \tag{3.2}$$

$$u_\nu + \beta u = q_2, \quad \text{on } \Gamma_R, \tag{3.3}$$

where Ω is a bounded domain with boundary $\partial\Omega$, u_ν is the outward normal derivative with respect to the boundary $\partial\Omega = \Gamma_N \cup \Gamma_R$, and β is a non-negative value.

To obtain the approximation solution of model problem (3.1)–(3.3), we consider the collocation method based on reproducing kernel approximation,

$$u^r = \sum_{I=1}^{N_p} \Psi_I(\mathbf{x}) u_I, \quad u^r \in V_k, \tag{3.4}$$

where the coefficients u_I will be obtained by strong form collocation method to be discussed in this section.

The collocation method can be regarded as the least-squares method with integration quadratures [24]. The least-squares method is to seek the solution $u^r \in V_k$ such that

$$E(u^r) = \min_{v \in V_k} E(v), \tag{3.5}$$

where

$$E(v) = \frac{1}{2} \left\{ \int_\Omega (\Delta v + f)^2 d\Omega + \int_{\Gamma_N} (v_\nu - q_1)^2 d\ell + \int_{\Gamma_R} (v_\nu + \beta v - q_2)^2 d\ell \right\}. \tag{3.6}$$

The minimization problem in (3.5) can be described equivalently in the following variational formulation

$$B(u^r, v) = F(v), \quad \forall v \in V_k, \tag{3.7}$$

where

$$B(u,v)=\int_{\Omega}\Delta u\Delta v d\Omega+\int_{\Gamma_N}u_\nu v_\nu d\ell+\int_{\Gamma_R}(u_\nu+\beta u)(v_\nu+\beta v)d\ell, \tag{3.8}$$

$$F(v)=-\int_{\Omega}f\Delta v d\Omega+\int_{\Gamma_N}q_1 v_\nu d\ell+\int_{\Gamma_R}q_2(v_\nu+\beta v)d\ell. \tag{3.9}$$

The integrals in (3.6), (3.8), and (3.9) can be numerically evaluated using quadrature rules, for example, the Newton-Cotes or the Gaussian quadrature rules

$$\int_{\Omega}g d\Omega \approx \widehat{\int_{\Omega}} g d\Omega=\sum_{ij}\alpha_{ij}g(Q_{ij}), \quad \forall Q_{ij}\in\Omega, \tag{3.10}$$

$$\int_{\Gamma_N}g d\ell \approx \widehat{\int_{\Gamma_N}} g d\ell=\sum_{i}\alpha_i^N g(Q_i^N), \quad \forall Q_i^N\in\Gamma_N, \tag{3.11}$$

$$\int_{\Gamma_R}g d\ell \approx \widehat{\int_{\Gamma_R}} g d\ell=\sum_{j}\alpha_j^R g(Q_j^R), \quad \forall Q_j^R\in\Gamma_R, \tag{3.12}$$

where α_{ij}, α_i^N and α_j^R are weights and Q_{ij}, Q_i^N and Q_j^R are integration points (collocation points).

The problem (3.5) involving quadrature approximation leads to a discrete problem. The collocation method is to seek the solution $\tilde{u}^r \in V_k$ such that

$$\hat{E}(\tilde{u}^r)=\min_{v\in V_k}\hat{E}(v), \tag{3.13}$$

where $\hat{E}(\cdot)$ denotes the quadrature approximation of $E(\cdot)$ in (3.6). This also can be described equivalently

$$\hat{B}(\tilde{u}^r,v)=\hat{F}(v), \quad \forall v\in V_k, \tag{3.14}$$

where $\hat{B}(\cdot,\cdot)$ and $\hat{F}(\cdot)$ denote the quadrature versions of $B(\cdot,\cdot)$ and $F(\cdot)$ in (3.8) and (3.9), respectively.

B. The Implementation Scheme

The detailed algorithm and the complexity of the reproducing kernel collocation method are given in this section.

The discrete functional $\hat{E}(v)$ in (3.13) can be further manipulated as

$$\hat{E}(v)=\frac{1}{2}\sum_{J=1}^{N_o}\alpha_J\{(\Delta v+f)(\xi_J)\}^2+\frac{1}{2}\sum_{J=1}^{N_a}\alpha_J^N\{(v_\nu-q_1)(\xi_J)\}^2 + \frac{1}{2}\sum_{J=1}^{N_b}\alpha_J^R\{((v_\nu+\beta v-q_2))(\xi_J)\}^2, \tag{3.15}$$

where ξ_J denotes the integration nodes (collocation points), $\alpha_J, \alpha_J^N, \alpha_J^R$ are the weights, N_o, N_a, and N_b are the number of integration nodes in Ω, Γ_N, and Γ_R, respectively. Since

$$v\in V_k, \quad V_k=\text{span}\{\Psi_1,\Psi_2,\ldots,\Psi_{N_p}\}, \tag{3.16}$$

the minimization of (3.15) leads to solving a linear system below,

$$-\sqrt{\alpha_J}\sum_{I=1}^{N_P}\Delta\Psi_I(\xi_J)u_I = \sqrt{\alpha_J}f(\xi_J), \quad \forall\xi_J \in \Omega, \tag{3.17}$$

$$\sqrt{\alpha_J^N}\sum_{I=1}^{N_P}\Psi_{I,\nu}(\xi_J)u_I = \sqrt{\alpha_J^N}q_1(\xi_J), \quad \forall\xi_J \in \Gamma_N, \tag{3.18}$$

$$\sqrt{\alpha_J^R}\sum_{I=1}^{N_P}\{\Psi_{I,\nu} + \beta\Psi_I\}(\xi_J)u_I = \sqrt{\alpha_J^R}\, q_2(\xi_J), \quad \forall\xi_J \in \Gamma_R. \tag{3.19}$$

The above equations can be written as

$$\mathbf{Ay} = \mathbf{b}, \tag{3.20}$$

where the entry of matrix $\mathbf{A}$ is given by

$$[\mathbf{A}]_{IJ} = \begin{cases} -\sqrt{\alpha_J}\Delta\Psi_I(\xi_J), & J = 1,\dots,N_o, \\ \sqrt{\alpha_J^N}\Psi_{I,\nu}(\xi_J), & J = N_o+1,\dots,N_o+N_a, \\ \sqrt{\alpha_J^R}\{\Psi_{I,\nu}+\beta\Psi_I\}(\xi_J), & J = N_o+N_a+1,\dots,N_o+N_a+N_b, \end{cases} \tag{3.21}$$

where $I = 1, 2, \dots, N_P$, N_P is the number of discrete points. The coefficient vector $\mathbf{y}$ is defined as

$$\mathbf{y} = [u_1, u_2, \dots, u_{N_P}]^{\mathrm{T}}, \tag{3.22}$$

and the component of vector $\mathbf{b}$ is given by

$$[\mathbf{b}]_J = \begin{cases} \sqrt{\alpha_J}f(\xi_J), & J = 1,\dots,N_o, \\ \sqrt{\alpha_J^N}q_1(\xi_J), & J = N_o+1,\dots,N_o+N_a, \\ \sqrt{\alpha_J^R}q_2(\xi_J), & J = N_o+N_a+1,\dots,N_o+N_a+N_b. \end{cases} \tag{3.23}$$

The matrix $\mathbf{A}$ in (3.20) is with dimension $N_c \times N_P$, the vector $\mathbf{y}$ is with dimension N_P, and the vector $\mathbf{b}$ is with dimension N_c, where $N_c = N_o + N_a + N_b$ is the total number of collocation points in $\Omega \cup \Gamma_N \cup \Gamma_R$.

To ensure an optimal solution, the number of collocation points N_c should be greater than the number of nodal points N_P and this leads to an over-determined system. A relationship for the choice of the values N_c and N_P will be discussed in Section IVA.

For solving the over-determined system (3.20), we use the following algorithm:

(i) Compute $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ and $\mathbf{c} = \mathbf{A}^{\mathrm{T}}\mathbf{b}$.
(ii) Compute the Cholesky decomposition $\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{LL}^{\mathrm{T}}$.
(iii) Solve $\mathbf{Lz} = \mathbf{c}$ for $\mathbf{z}$ by backward substitution (BS).
(iv) Solve $\mathbf{L}^{\mathrm{T}}\mathbf{y} = \mathbf{z}$ for $\mathbf{y}$ by forward substitution (FS).

Remark 3.1. For solving the over-determined system, we may use QR decomposition or singular value decomposition (SVD), however, the costs for these two decompositions are much higher than the Cholesky decomposition.

The total operation counts for RKCM consists of (1) forming a linear system (3.20) and (2) solving the over-determined system by the algorithm described above. For forming a matrix **A** and vector **b** in a 1D Poisson's problem with Dirichlet and Neumann boundary conditions, the operation count is

$$\begin{aligned}\text{op}:\quad & \kappa\cdot\{s^3+(2\kappa+1)s^2+s+1\}+\kappa\cdot\{3s^3+(8\kappa+4)s^2+3s+2\}\\ &+\kappa\cdot(N_c-2)\cdot\{6s^3+(20\kappa+12)s^2+6s+4\},\end{aligned}\tag{3.24}$$

where $s = p + 1$. The operation count in forming $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ and $\mathbf{A}^{\mathrm{T}}\mathbf{b}$, and using the Cholesky decomposition with substitutions to find **y** is

$$\text{op}:\quad N_c\cdot N_P^2/2+N_c\cdot N_P+N_P^3/6+N_P^2/2+N_P^2/2.\tag{3.25}$$

By considering $\kappa \approx s = p + 1$ and $N_c = 4 \cdot N_P$, the total operation count is

$$\text{op}:\quad O\big((104\cdot N_P-38)\cdot(p+1)^4\big)+O\big(13\cdot N_P^3/6\big).\tag{3.26}$$

IV. ANALYSIS OF RKCM

In this section, we first discuss the convergence property of RKCM for the model problem in (3.1)–(3.3). Considering that the minimization problem is equivalent to variational formulations, the implementation scheme is based on the minimization problem, whereas the error analysis is based on the variational formulation.

The error analysis consists of two steps: the continuous formulation without any quadrature rules involved, and the formulation with the quadrature rules. For the stability analysis, the inverse inequality stated in Theorem 2.1 is used.

A. Convergence

Denote the space

$$H=\{v|v\in H^1(\Omega),\Delta v\in L^2(\Omega)\},\tag{4.1}$$

where H^1 is the Sobolve space, accompanied with the norm

$$|||v|||=\big\{\|v\|_{1,\Omega}^2+\|\Delta v\|_{0,\Omega}^2+\|v_\nu\|_{0,\Gamma_N}^2+\|v_\nu+\beta v\|_{0,\Gamma_R}^2\big\}^{\frac{1}{2}}.\tag{4.2}$$

The following Lemma is used for the derivation later.

Lemma 4.1. *Assume a quasi-uniform points distribution. For $v \in V_k$, there exist the following inverse inequalities*

$$\|v\|_{\ell,\Omega}\le C\sqrt{\kappa}a^{-\ell}p^{2\ell}\ \|v\|_{0,\Omega},\tag{4.3}$$

$$\|v\|_{\ell,\Gamma}\le C\sqrt{\kappa}a^{-\ell}p^{2\ell}\|v\|_{0,\Gamma}\le C\sqrt{\kappa}a^{-\ell}p^{2\ell}\|v\|_{1,\Omega},\tag{4.4}$$

$$\|v_\nu\|_{\ell,\Gamma}\le C\sqrt{\kappa}a^{-(\ell+1)}p^{2\ell+2}\|v\|_{0,\Gamma}\le C\sqrt{\kappa}a^{-(\ell+1)}p^{2\ell+2}\|v\|_{1,\Omega},\tag{4.5}$$

for $\ell \ge 1$, where C is a generic constant.

Note that the detailed proof for (4.3) is given in Section IID, see Theorem 2.1, and the inequalities (4.4) and (4.5) can be derived in a similar manner.

In the first step of analysis, we need the following Lemmas.

Lemma 4.2. *There exists an inequality between the bilinear form and the Sobolev one norm*

$$B(v, v) \geq C\|v\|_{1,\Omega}^2, \quad \forall v \in V_k, \tag{4.6}$$

where C is a positive generic constant.

Proof. Using the integration by parts, we have

$$\begin{aligned} |v|_{1,\Omega}^2 &= \int_\Omega (\nabla v)^2 d\Omega = -\int_\Omega v\Delta v d\Omega + \int_{\partial\Omega} v_\nu v d\ell \\ &= -\int_\Omega v\Delta v d\Omega + \int_{\Gamma_N} v_\nu v d\ell + \int_{\Gamma_R} v_\nu v d\ell \end{aligned} \tag{4.7}$$

where ν is the outward normal. The following bounds exist

$$\|v\|_{0,\Gamma_N} \leq C\|v\|_{1,\Omega}, \quad \forall v \in V_k, \tag{4.8}$$

$$\|v\|_{0,\Gamma_R} \leq C\|v\|_{1,\Omega}, \quad \forall v \in V_k, \tag{4.9}$$

and

$$\left|\int_\Omega v\Delta v d\Omega\right| \leq C\|\Delta v\|_{0,\Omega}\|v\|_{0,\Omega}, \tag{4.10}$$

$$\left|\int_{\Gamma_N} v_\nu v d\ell\right| \leq C\|v_\nu\|_{0,\Gamma_N}\|v\|_{0,\Gamma_N}, \tag{4.11}$$

$$\left|\int_{\Gamma_R} v_\nu v d\ell\right| \leq C\|v_\nu\|_{0,\Gamma_R}\|v\|_{0,\Gamma_R}. \tag{4.12}$$

It follows from (4.7) that

$$|v|_{1,\Omega}^2 \leq C\{\|\Delta v\|_{0,\Omega} + \|v_\nu\|_{0,\Gamma_N} + \|v_\nu\|_{0,\Gamma_R}\}\|v\|_{1,\Omega}. \tag{4.13}$$

Using Poincare's inequality,

$$\|v\|_{1,\Omega} \leq C|v|_{1,\Omega}, \tag{4.14}$$

Eq. (4.13) becomes

$$\|v\|_{1,\Omega} \leq C\{\|\Delta v\|_{0,\Omega} + \|v_\nu\|_{0,\Gamma_N} + \|v_\nu\|_{0,\Gamma_R}\}. \tag{4.15}$$

Moreover, we have

$$\begin{aligned} \|v\|_{1,\Omega}^2 &\leq C_0\left\{\|\Delta v\|_{0,\Omega}^2 + \|v_\nu\|_{0,\Gamma_N}^2 + \|v_\nu\|_{0,\Gamma_R}^2\right\} \\ &\leq C_1\left\{\|\Delta v\|_{0,\Omega}^2 + \|v_\nu\|_{0,\Gamma_N}^2 + \|v_\nu + \beta v\|_{0,\Gamma_R}^2\right\} = C_1 B(v, v), \end{aligned} \tag{4.16}$$

where β is a positive number. Hence

$$B(v, v) \geq C\|v\|_{1,\Omega}^2. \tag{4.17}$$

This proves (4.6). ∎

Lemma 4.3. *There exist two inequalities*

$$B(u, v) \leq C_1 |||u|||\ |||v|||, \quad \forall v \in V_k, \tag{4.18}$$

$$B(v, v) \geq C_0 |||v|||^2, \quad \forall v \in V_k, \tag{4.19}$$

where C_0 and C_1 are two positive generic constants.

Proof. First, from definitions (3.8) and (4.2), we have

$$\begin{aligned} B(u, v) &\leq C\left\{\|\Delta u\|_{0,\Omega}^2 + \|u_\nu\|_{0,\Gamma_N}^2 + \|u_\nu + \beta u\|_{0,\Gamma_R}^2\right\}^{\frac{1}{2}} \\ &\quad \times \left\{\|\Delta v\|_{0,\Omega}^2 + \|v_\nu\|_{0,\Gamma_N}^2 + \|v_\nu + \beta v\|_{0,\Gamma_R}^2\right\}^{\frac{1}{2}} \\ &\leq C_1\left\{\|u\|_{1,\Omega}^2 + \|\Delta u\|_{0,\Omega}^2 + \|u_\nu\|_{0,\Gamma_N}^2 + \|u_\nu + \beta u\|_{0,\Gamma_R}^2\right\}^{\frac{1}{2}} \\ &\quad \times \left\{\|v\|_{1,\Omega}^2 + \|\Delta v\|_{0,\Omega}^2 + \|v_\nu\|_{0,\Gamma_N}^2 + \|v_\nu + \beta v\|_{0,\Gamma_R}^2\right\}^{\frac{1}{2}} \\ &= C_1 |||u|||\,|||v|||. \end{aligned} \tag{4.20}$$

Second, by using (4.6), we have

$$\begin{aligned} B(v, v) &= \frac{1}{2}B(v, v) + \frac{1}{2}B(v, v) \\ &\geq C\|v\|_{1,\Omega}^2 + \frac{1}{2}\left\{\|\Delta v\|_{0,\Omega}^2 + \|v_\nu\|_{0,\Gamma_N}^2 + \|v_\nu + \beta v\|_{0,\Gamma_R}^2\right\} \\ &\geq C_0 |||v|||^2, \end{aligned} \tag{4.21}$$

where $C_0 = \min\{\frac{1}{2}, C\}$. The result (4.19) is obtained. ∎

For the second step for analysis, the following two Lemmas are needed.

Lemma 4.4. *For a quadrature rule with order γ, and for $v \in V_k$, there exist the bounds,*

$$\left|\left(\int_\Omega - \widehat{\int_\Omega}\right)(\Delta v)^2\right| \leq C_1(\kappa, p)\hbar^{(\gamma+1)}a^{-(\gamma+3)}\|v\|_{1,\Omega}^2, \tag{4.22}$$

$$\left|\left(\int_{\Gamma_N} - \widehat{\int_{\Gamma_N}}\right)(v_\nu)^2\right| \leq C_2(\kappa, p)\hbar^{(\gamma+1)}a^{-(\gamma+3)}\|v\|_{1,\Omega}^2, \tag{4.23}$$

$$\left|\left(\int_{\Gamma_R} - \widehat{\int_{\Gamma_R}}\right)(v_\nu + \beta v)^2\right| \leq C_3(\kappa, p)\hbar^{(\gamma+1)}a^{-(\gamma+3)}\|v\|_{1,\Omega}^2, \tag{4.24}$$

where $\hbar$ denotes the maximal spacing of the collocation points and C_i are the constants dependent on κ and p.

Proof. Let $g = (v_\nu)^2$. By using the Mean Value Theorem for Integrals and the inverse inequalities stated in Lemma 4.1, we have

$$\left|\left(\int_{\Gamma_N} - \widehat{\int_{\Gamma_N}}\right)(v_\nu)^2\right| = \left|\int_{\Gamma_N} g - \widehat{\int_{\Gamma_N}} g\right| = \left|\int_{\Gamma_N}(g - \hat{g})\right| \le C\hbar^{\gamma+1}|g|_{\gamma+1,\Gamma_N} \tag{4.25}$$

where

$$\begin{aligned} |g|_{\gamma+1,\Gamma_N} &= |(v_\nu)^2|_{\gamma+1,\Gamma_N} \le C\sum_{i=0}^{\gamma+1}|v_\nu|_{\gamma+1-i,\Gamma_N}|v_\nu|_{i,\Gamma_N} \\ &\le C\sum_{i=0}^{\gamma+1}\sqrt{\kappa}a^{-(\gamma+1-i+1)}p^{2\gamma+2-2i+2}\|v\|_{1,\Omega}\cdot\sqrt{\kappa}a^{-(i+1)}p^{2i+2}\|v\|_{1,\Omega} \\ &\le C\kappa a^{-(\gamma+3)}p^{2(\gamma+3)}\|v\|^2_{1,\Omega}. \end{aligned} \tag{4.26}$$

Combining (4.25) and (4.26), we obtain

$$\begin{aligned} \left|\left(\int_{\Gamma_N} - \widehat{\int_{\Gamma_N}}\right)(v_\nu)^2\right| &\le C\hbar^{\gamma+1}\kappa a^{-(\gamma+3)}p^{2(\gamma+3)}\|v\|^2_{1,\Omega} \\ &\le C_1(\kappa, p)\hbar^{\gamma+1}a^{-(\gamma+3)}\|v\|^2_{1,\Omega}. \end{aligned} \tag{4.27}$$

The other inequalities in (4.22) and (4.24) can be derived in a similar manner. ■

Lemma 4.5. *Suppose that Lemmas 4.3 and 4.4 hold, we choose $\hbar$ to satisfy*

$$\hbar^{\gamma+1}a^{-(\gamma+3)} = o(1), \tag{4.28}$$

then there exist

$$\hat{B}(u, v) \le C_1|||u|||\ |||v|||, \quad \forall v \in V_k, \tag{4.29}$$

$$\hat{B}(v, v) \ge C_0|||v|||^2, \quad \forall v \in V_k, \tag{4.30}$$

where C_0 and C_1 are positive generic constants.

Proof. We prove (4.30) only; Eq. (4.29) is easy to obtain.
It can be derived from Lemmas 4.3 and 4.4 that

$$\begin{aligned} \hat{B}(v, v) &\ge B(v, v) - C\hbar^{\gamma+1}a^{-(\gamma+3)}\|v\|^2_{1,\Omega} \\ &\ge C_0|||v|||^2 - C\hbar^{\gamma+1}a^{-(\gamma+3)}\|v\|^2_{1,\Omega} \\ &\ge C_0\left\{\left(1 - \frac{C}{C_0}\hbar^{\gamma+1}a^{-(\gamma+3)}\right)\|v\|^2_{1,\Omega} + \|\Delta v\|_{0,\Omega} + \|v_\nu\|^2_{0,\Gamma_N} + \|v_\nu + \beta v\|^2_{0,\Gamma_R}\right\} \\ &\ge \frac{C_0}{2}|||v|||^2, \end{aligned} \tag{4.31}$$

provided that

$$\frac{C}{C_0}\hbar^{\gamma+1}a^{-(\gamma+3)} \le \frac{1}{2}. \tag{4.32}$$

This proves (4.30). ■

Using two important Lemmas 4.3 and 4.5, we have an optimal error estimation for the solution $\tilde{u}^r$.

Theorem 4.1. *Under the conditions in Lemma 4.5, the solution of the reproducing kernel collocation method (RKCM) has the bound*

$$|||u - \tilde{u}^r||| \le C \inf_{v\in V_k} |||u - v|||, \tag{4.33}$$

where C is a constant independent on a and N_P.

Proof. From Lemmas 4.3 and 4.4, we have

$$\hat{B}(u, v) \le \hat{F}(v) + C\hbar^{\gamma+1}a^{-(\gamma+3)}\|v\|^2_{1,\Omega}, \quad \forall v \in V_k, \tag{4.34}$$

and the solution $\tilde{u}^r$ satisfies

$$\hat{B}(\tilde{u}^r, v) = \hat{F}(v), \quad \forall v \in V_k. \tag{4.35}$$

Therefore,

$$\hat{B}(u - \tilde{u}^r, v) \le C\hbar^{\gamma+1}a^{-(\gamma+3)}\|v\|^2_{1,\Omega}, \quad \forall v \in V_k. \tag{4.36}$$

Let $w = \tilde{u}^r - v \in V_k$, and using the results stated in Lemma 4.5 and (4.36), it follows that

$$\begin{aligned} C_0|||w|||^2 &\le \hat{B}(\tilde{u}^r - v, w) = \hat{B}(\tilde{u}^r - u + u - v, w) \\ &\le \hat{B}(u - v, w) + \hat{B}(u - \tilde{u}^r, w) \\ &\le C_1|||u - v|||\ |||w||| + C\hbar^{\gamma+1}a^{-(\gamma+3)}\|w\|^2_{1,\Omega} \\ &\le C_1|||u - v|||\ |||w||| + C\hbar^{\gamma+1}a^{-(\gamma+3)}|||w|||^2. \end{aligned} \tag{4.37}$$

This leads to

$$|||\tilde{u}^r - v||| = |||w||| \le \frac{C_1}{C_0 - C \times o(1)}|||u - v||| =: C_2|||u - v|||, \tag{4.38}$$

and consequently

$$|||u - \tilde{u}^r||| \le |||u - v||| + |||v - \tilde{u}^r||| \le (1 + C_2)|||u - v|||. \tag{4.39}$$

The result (4.33) is obtained. ■

Remark 4.1. Recall the condition between the collocation points and the nodal points of the RK shape function described in Lemma 4.5

$$\hbar \approx o(a^{1+\frac{2}{\gamma+1}}). \tag{4.40}$$

Since $a = O(h)$, where h is nodal distance, we have

$$\hbar \approx o(h^2), \quad \text{when } \gamma = 1, \tag{4.41}$$

$$\hbar \approx o(h^{\frac{5}{3}}), \quad \text{when } \gamma = 2, \tag{4.42}$$

$$\hbar \approx o(h^{\frac{3}{2}}), \quad \text{when } \gamma = 3, \tag{4.43}$$

$$\vdots \qquad \vdots$$

$$\hbar \approx o(h), \quad \text{very high order.} \tag{4.44}$$

It can be seen from (4.41)–(4.44) that $0 < \hbar \le h \ll 1$. This means that according to (4.40), the density of collocation points is selected much denser than the density of the nodal points if low order integration rules are used, i.e., $N_c > N_P$.

Corollary 4.1. *According to Theorem 4.1 and Lemma 2.2, the error of RKCM solution is bounded by*

$$\begin{aligned} |||u - \tilde{u}^r||| &\le C\left\{\|u - v\|_{2,\Omega} + \|(u - v)_\nu\|_{0,\Gamma_N} + \|(u - v)_\nu + \beta(u - v)\|_{0,\Gamma_R}\right\}, \\ &\le C\kappa a^{p-1}|u|_{p+1,\Omega}. \end{aligned} \tag{4.45}$$

where p is reproducing degree and C is generic constant.

Remark 4.2. In the proposed collocation approach, it is important to note that the solution does not converge when reproducing degree $p = 1$ is used for the Poisson problem. Recall the reproducing conditions for two-dimension

$$\sum_{I=1}^{N_P} \Psi_I(\mathbf{x})\, x_I^i y_I^j = x^i y^j, \quad i + j = 0, 1, \ldots, p. \tag{4.46}$$

According to (4.45), the reproducing degree p should be at least 2.

To meet reproducing conditions in (4.46), any position $\mathbf{x} \in \Omega$ needs to be covered by 6 non-collinear RK kernel functions for $p = 2$ and be covered by 10 non-collinear RK kernel functions for $p = 3$. For uniform points distribution, support size a should be chosen as

$$a \ge 3h, \quad \text{for } p = 2, \tag{4.47}$$

$$a \ge 4h, \quad \text{for } p = 3, \tag{4.48}$$

where h is the nodal distance.

B. Stability

In this section, we perform the stability analysis for RKCM. For analysis purposes, we use different norms defined as follows, where they are related to the bilinear forms defined in Section III.

Define two norms

$$\|v\|_E = \left\{\|\Delta v\|_{0,\Omega}^2 + \|v_\nu\|_{0,\Gamma_N}^2 + \|v_\nu + \beta v\|_{0,\Gamma_R}\right\}^{\frac{1}{2}} = B(v,v)^{\frac{1}{2}}. \tag{4.49}$$

and

$$\overline{\|v\|}_E = \left\{\widehat{\int_\Omega}(\Delta v)^2 d\Omega + \widehat{\int_{\Gamma_N}}(v_\nu)^2 d\ell + \widehat{\int_{\Gamma_R}}(v_\nu + \beta v)^2 d\ell\right\}^{\frac{1}{2}} = \hat{B}(v,v)^{\frac{1}{2}}. \tag{4.50}$$

The norm discrete $\overline{\|v\|}_E^2$ can be expressed as

$$\overline{\|v\|}_E^2 = \hat{B}(v,v) = \mathbf{y}^{\mathrm{T}}\mathbf{G}\mathbf{y} := \mathbf{y}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{y}, \tag{4.51}$$

where $\mathbf{A}$ and $\mathbf{y}$ are defined in (3.21) and (3.22), respectively.

Lemma 4.6. *Following* (4.49) *and* (4.50)*, there exists a relationship*

$$C_1\|v\|_E \le \overline{\|v\|}_E \le C_2\|v\|_E, \tag{4.52}$$

where C_1 and C_2 are positive constants.

Proof. First, we have

$$\|v\|_E - \overline{\|v\|}_E \le \|v - \hat{v}\|_E \le C\hbar^{\gamma+1}\|D^{\gamma+1}v\|_E \le C_0\hbar^{\gamma+1}a^{-(\gamma+1)}\|v\|_E, \tag{4.53}$$

wherein the following relation is used:

$$\int_\Omega g - \widehat{\int_\Omega} g = \int_\Omega g - \int_\Omega \hat{g} = \int_\Omega (g - \hat{g}). \tag{4.54}$$

It follows that

$$\left(1 - C_0\hbar^{\gamma+1}a^{-(\gamma+1)}\right)\|v\|_E \le \overline{\|v\|}_E. \tag{4.55}$$

Assume $\hbar^{\gamma+1}a^{-(\gamma+1)} = o(1)$, and denote $C_1 = (1 - C_0\hbar^{\gamma+1}a^{-(\gamma+1)}) > 0$.

Similarly, we obtain

$$\overline{\|v\|}_E - \|v\|_E \le \|\hat{v} - v\|_E = \|v - \hat{v}\|_E \le C_0\hbar^{\gamma+1}a^{-(\gamma+1)}\|v\|_E. \tag{4.56}$$

Therefore, we have

$$\overline{\|v\|}_E \le (1 + C_0\hbar^{\gamma+1}a^{-(\gamma+1)})\|v\|_E =: C_2\|v\|_E. \tag{4.57}$$

This proves (4.52). ∎

Lemma 4.7. *For the norm defined in* (4.49), *there exists a relationship*

$$C_3\|v\|_{2,\Omega} \leq \|v\|_E \leq C_4\|v\|_{2,\Omega}, \tag{4.58}$$

where C_3 and C_4 are positive constants independent of a and N_P.

In the following, the bound for condition number is given, and it is inversely proportional to the square of the support size a.

Theorem 4.2. *Under the conditions in lemmas 4.6 and 4.7, there exist*

$$\lambda_{\min}(\mathbf{A}^{\mathrm{T}}\mathbf{A}) \geq \frac{C_a\|v\|_{0,\Omega}^2}{\mathbf{y}^{\mathrm{T}}\mathbf{y}}, \tag{4.59}$$

$$\lambda_{\max}(\mathbf{A}^{\mathrm{T}}\mathbf{A}) \leq \frac{C_b\|v\|_{2,\Omega}^2}{\mathbf{y}^{\mathrm{T}}\mathbf{y}}, \tag{4.60}$$

where C_a and C_b are positive constants. Furthermore, there exist

$$\text{Cond}\,(\mathbf{A}) \leq C_d a^{-2}. \tag{4.61}$$

where $C_d = C_d(\kappa, p)$ is constant dependent on κ and p.

Proof. Since $\mathbf{y}^{\mathrm{T}}\mathbf{G}\mathbf{y} = \overline{\|v\|}_E^2$, and using the Rayleigh-Ritz Theorem, we have

$$\lambda_{\min}(\mathbf{G}) = \min\frac{\mathbf{y}^{\mathrm{T}}\mathbf{G}\mathbf{y}}{\mathbf{y}^{\mathrm{T}}\mathbf{y}} \geq \min\frac{C_1\|v\|_E^2}{\mathbf{y}^{\mathrm{T}}\mathbf{y}} \geq \min\frac{C_3\|v\|_{0,\Omega}^2}{\mathbf{y}^{\mathrm{T}}\mathbf{y}}, \tag{4.62}$$

$$\lambda_{\max}(\mathbf{G}) = \max\frac{\mathbf{y}^{\mathrm{T}}\mathbf{G}\mathbf{y}}{\mathbf{y}^{\mathrm{T}}\mathbf{y}} \leq \max\frac{C_2\|v\|_E^2}{\mathbf{y}^{\mathrm{T}}\mathbf{y}} \leq \max\frac{C_4\|v\|_{2,\Omega}^2}{\mathbf{y}^{\mathrm{T}}\mathbf{y}}. \tag{4.63}$$

Using the inverse inequality (2.47), the condition number for matrix $\mathbf{G}$ is given by

$$\text{Cond}(\mathbf{G}) = \frac{\lambda_{\max}(\mathbf{G})}{\lambda_{\min}(\mathbf{G})} \leq \frac{C_3\|v\|_{2,\Omega}^2}{C_4\|v\|_{0,\Omega}^2} \leq \frac{C_5\kappa a^{-4}p^8\|v\|_{0,\Omega}^2}{C_4\|v\|_{0,\Omega}^2} \leq C\kappa a^{-4}p^8. \tag{4.64}$$

Since

$$\text{Cond}(\mathbf{G}) = \text{Cond}(\mathbf{A}^{\mathrm{T}}\mathbf{A}) = \{\text{Cond}(\mathbf{A})\}^2, \tag{4.65}$$

therefore

$$\text{Cond}(\mathbf{A}) \leq C\kappa a^{-2}p^4 =: C_d(\kappa, p)a^{-2}. \tag{4.66}$$

This proves (4.61). ∎

Remark 4.3. For uniform points distribution, κ is selected close to p so that the constant C_d in (4.61) is bounded in a reasonable range. The condition number increases as the model is refined, and it shows that RKCM is well-conditioned like the finite element method (FEM). The rate of condition number of FEM is of -2, i.e., $O(h^{-2})$, where h denotes the maximal mesh size.

V. NUMERICAL EXPERIMENTS

In this section, we present two numerical tests for one- and two-dimensional model problems to validate the theoretical analysis given in Sections IVA and IVB.

Since the support size a is proportional to the nodal distant h, it can be seen from Corollary 4.1 that

$$|||u - \tilde{u}^r||| \approx O(h^{p-1}), \tag{5.1}$$

where p has to be greater than one. Furthermore, we have the following estimations

$$\|u - \tilde{u}^r\|_{0,\Omega} \approx O(h^{p+1}), \tag{5.2}$$

$$\|u - \tilde{u}^r\|_{1,\Omega} \approx O(h^{p}), \tag{5.3}$$

$$\|u - \tilde{u}^r\|_{2,\Omega} \approx O(h^{p-1}). \tag{5.4}$$

By choosing support size $a = (p + \delta)h$, where $\delta > 0$, the bound of condition number is

$$\text{Cond}(\mathbf{A}) \leq C_d(\kappa, p)a^{-2} = \frac{C\kappa p^4}{(p+\delta)^2} \cdot h^{-2} \approx O(h^{-2}). \tag{5.5}$$

In numerical results given in the following examples, we illustrate how error and condition number are related to discretization.

A. Example I

Consider a one-dimensional (1D) model problem of the form

$$u''(x) = e^x \quad \text{in} \quad \Omega, \tag{5.6}$$

$$u(0) = 1, \tag{5.7}$$

$$u(1) = e, \tag{5.8}$$

where $\Omega = \{x \,|\, 0 < x < 1\}$. The analytical solution is $u(x) = e^x$. We use the reproducing degree $p = 1, 2, 3$, the number of nodal points $N_P = 6, 8, \ldots, 16$, and the number of collocation points $N_c = 24, 32, \ldots, 64$. Equally spaced collocation points and nodal points are adopted. We define $h = 1/N_P$ and choose $a = (p + 1)h$ as support size of RK shape function. Theoretical analysis, see Remark 4.1, states that the number of collocation points should be greater than the number of nodal points. Here, the total number of collocation points used is four times the number of nodal points.

The curves of the error in Sobolev zero norm with various level of nodal refinements are shown in Fig. 2(a). It is worth noting that the case of linear basis, $p = 1$, does not converge at all, and for $p = 2, 3$, the rates of convergence agree well with the analytical predication. The condition numbers are shown in Fig. 2(b). It can be seen that these numerical rates are in fact better than the analytical prediction of -2 as given in Section IVB.

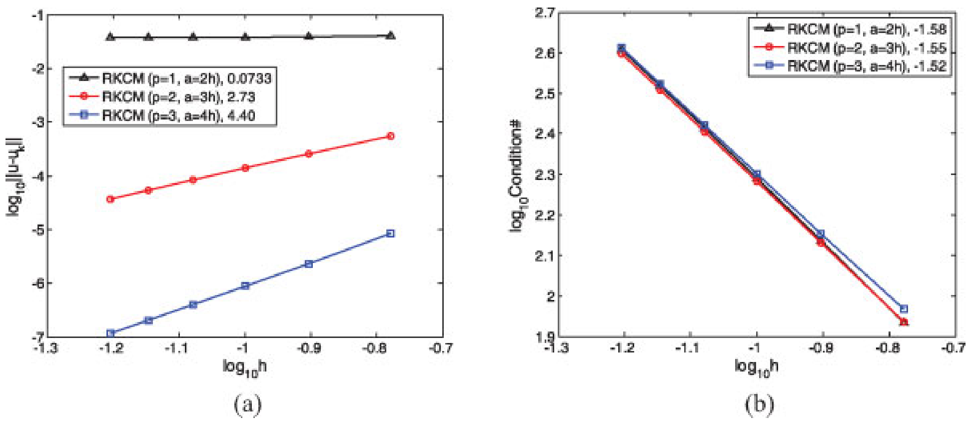

FIG. 2. The convergence and the condition number of RKCM with $p = 1, 2, 3$ for the 1D model problem. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

B. Example II

Consider a two-dimensional (2D) model problem as follows

$$\Delta u = f(x, y) \quad \text{in } \Omega, \tag{5.9}$$

$$u = 1 \quad \text{on } x = 0, \tag{5.10}$$

$$u = 1 \quad \text{on } y = 0, \tag{5.11}$$

$$u = e^{y} \quad \text{on } x = 1, \tag{5.12}$$

$$u = e^{x} \quad \text{on } y = 1, \tag{5.13}$$

where $\Omega = \{(x, y) |\ 0 < x < 1,\ 0 < y < 1\}$. The function $f(x, y)$ is given as $f(x, y) = (x^2 + y^2)\, e^{xy}$. The analytical solution is $u(x, y) = e^{xy}$. The cases for reproducing degree $p = 1, 2, 3$ are considered, and we use the number of nodal points $N_P = 6^2, 8^2, \ldots, 16^2$, and the number of collocation points $N_c = 24^2, 32^2, \ldots, 64^2$. We define $h = 1/\sqrt{N_P}$ and choose $a = (p + 1)h$ as support size.

A convergence behavior similar to Example I is shown in Fig. 3(a), and again the case with $p = 1$ the solution does not converge. The condition numbers have rate close to -2 as predicted by the analysis given in Section IVB.

VI. CONCLUSIONS

Reproduction kernel approximation has been widely used in the arena of "meshfree method" under the Galerkin weak framework. Despite its mathematical robustness in solving PDE with versatility in adjusting smoothness and locality in the approximation, performing adaptive refinement in discretization, and in dealing with fracture and large deformation problems, the need for domain integration and treatment of Dirichlet boundary conditions adds considerable complexity in the numerical calculation. Alternatively, strong form collocation has been introduced in conjunction with radial basis functions for numerical solution of PDE. This approach eliminates the need for quadrature rule and offers a straightforward imposition of boundary conditions. Nevertheless, the

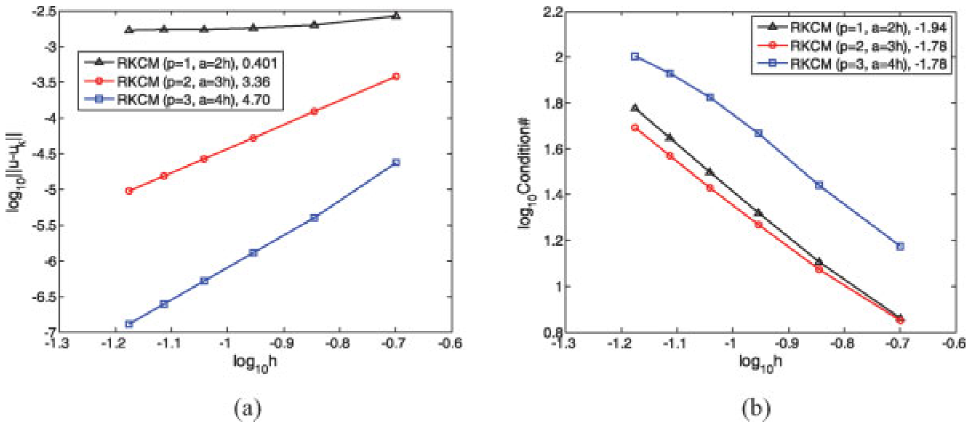

FIG. 3. The convergence and the condition number of RKCM with $p = 1, 2, 3$ for the 2D model problem. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

nonlocality of radial basis functions yields a full and ill-conditioned discrete system that limits it application to simple geometry, small-scale calculation. Thus, the employment of reproducing kernel approximation with compact support in conjunction with the strong form collocation offers complementary advantages of the above-mentioned methods.

The objective of this article is to offer a mathematical analysis of reproducing kernel collocation method. Specifically, we provide an inverse inequality of reproducing kernel approximation, the optimal estimation, the bounds for condition number, and the optimal relationship for the number of nodal points and the number of collocation points. The theoretical analysis shows that the degree of monomial basis functions in the reproducing kernel approximation has to be greater than one for convergence in this approach. This result is different from the weak form type reproducing kernel particle method, in that monomial degree of zero is sufficient for convergence. In addition, the condition number of this approach increases algebraically, indicating a stability character similar to the finite element method and thus yields a significant improvement in stability over the radial basis collocation method.

Several numerical examples have been given to validate the agreement between theoretical prediction and numerical solution in terms of convergence, stability, especially the necessary condition of the minimum monomial reproducing order for convergence.

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